3
Random walks in groups

In this section we wish to remind the reader of the definition of a random walk and to present two propositions. These are the results that the subsequent sections try to mimic in a non-random setting.

By a *(right) random walk* on a finitely generated group Γ we mean a probability measure p on Γ. We shall imagine that a particle is placed at the identity of Γ and that it moves with discrete time. The probability of moving from a point x to a point y (that is, the probability that $x^{-1}y$ acts on the right) is given by

$$p(x,y) = p(x^{-1}y).$$

The numbers $p(x,y)$ can be thought as making up a (possibly infinite) matrix P. The (x,y)-entry of P^n will be denoted by $p^{(n)}(x,y)$ and it is the probability of moving from x to y in exactly n steps. The random walk is said to be *irreducible* if, for every $x,y \in \Gamma$, there exists a positive integer n such that $p^{(n)}(x,y) > 0$.

We set $\Omega = \Gamma^{\mathbb{N}}$ and we note $Z_i : \Omega \to \Gamma$ the projection on the i^{th} coordinate. The space Ω is equipped with the Borel σ-algebra for the product topology (each factor with the discrete topology) and the probability measure

$$\mathbb{P}[Z_0 = x_0, \ldots, Z_n = x_n] = \delta_{id}(x_0)p(x_0,x_1)\ldots p(x_{n-1},x_n).$$

The *first moment* of p with respect to a word norm (see Section 6.1) $|\cdot|$ on Γ is

$$M_1(p) = \int_\Omega |Z_1| \; d\mathbb{P} = \sum_n n \left(\sum_{|y|=n} p(y) \right).$$

If the first moment is finite for some (and thus, by Proposition 6.1.3, every) word norm, p is said to *have finite first moment*.

The two following propositions can be found in (13), pages 88 and 125.

Proposition 3.0.7. *If the random walk p on Γ has finite first moment, then there exists a constant m such that, $\mathbb{P}$-almost surely,*

$$\lim_{n\to\infty} \frac{1}{n}|Z_n| = m.$$

Proposition 3.0.8. *If p is an irreducible random walk on a non-amenable group Γ then $m > 0$.*

Remark 3.0.9. *While the actual value of m depends on the choice of $|\cdot|$, the condition $m > 0$ does not, by Proposition 6.1.3.*