5
The probabilistic model

This is the main chapter of this dissertation. We introduce the probabilistic model (a non-random walk in Γ), prove some basic properties and the positive linear growth.

5.1
Definition

We will now interpret the dynamics of F as a walk in Γ. We will study how F transfers the measure μ from S_0 to other fundamental regions of the form σS_0, with $\sigma \in \Gamma$. We set

$$\mu(\sigma) = \mu(S_0 \cap F^{-1}(\sigma S_0)).$$

As mentioned earlier, we allow ourselves to identify S and S_0.

Remark 5.1.1. *μ should not be seen as a* random *walk in Γ. For instance, $\mu(S_0 \cap F^{-1}(\sigma_1 S_0) \cap F^{-2}(\sigma_1\sigma_2 S_0))$ is not necessarily equal to $\mu(\sigma_1)\mu(\sigma_2)$. One can think of this as a lack of indepence (or existence of correlation) between the iterates of F.*

We set Q to be the elements of Γ that actually "act" in this walk, that is,

$$Q = \{\sigma \in \Gamma \ : \ \mu(\sigma) > 0\}.$$

There is also a natural map $g : \mathbb{D} \to \Gamma$ that associates to a point x of $\mathbb{D}$ the deck transformation σ such that x lies in σS_0. We can then associate, to (almost) every point x of $\mathbb{D}$, a sequence $w(x)$ in $Q^{\mathbb{N}}$ that tells us how x moves among the fundamental regions:

$$w(x) = (g(F^{n-1}(x_0))^{-1} g(F^n(x_0)))_{n>0}.$$

This sequence is constructed in such a way that $w_1 \cdot \ldots \cdot w_n(x) = g(F^n(x))$. If $T : Q^{\mathbb{N}} \to Q^{\mathbb{N}}$ is the standard shift map, it is imediate that $w \circ f = T \circ w$. Hence, by equipping $Q^{\mathbb{N}}$ with the probability measure $\mathbb{P} = w_*\mu$ we get a measurable conjugation between T and f. In particular, T becomes ergodic.

5.2 Basic properties

We begin by proving that Q contains a non-trivial element of Γ and is finite.

Lemma 5.2.1. *There exists $\sigma \in Q$ such that $\sigma \neq id$.*

Proof. Since $\mu(\Gamma) = 1$ and Γ is countable, Q cannot be empty. Suppose that $Q = \{id\}$, which implies $\mu(id) = 1$. Hence, for almost every x_0 in S_0, the image $F(x_0)$ is also in S_0. Poincaré's recurrence theorem (Theorem A.1.1), applied to $F|_{S_0}$, tells us that almost every point of S_0 is recurrent.

The idea is to use this fact to construct compactly supported perturbations of F with periodic points. Fix x_0 recurrent by F and $\epsilon_0 > 0$. If d is the hyperbolic metric, the absence of fixed points for F (by construction) implies that $d(y, F(y))$ attains a strictly positive infimum δ on the closed ball $\overline{B_{\epsilon_0}}$, of center x_0 and radius ϵ_0. Consequently, a map $\Phi : \mathbb{D} \to \mathbb{D}$ which coincides with F outside $\overline{B_{\epsilon_0}}$ and such that $d(F(y), \Phi(y)) < \delta$ for every y cannot have fixed points.

Let $\epsilon < \min \epsilon_0, \delta$ and N a positive integer such that $d(x_0, F^N(x_0)) < \epsilon$. We choose a neighborhood V of x_0 contained in $\overline{B_{\epsilon_0}}$ such that $F^N(x_0)$ is in V but $F^i(x_0)$, for $0 < i < N$ is not. There exists a diffeomorphism f isotopic to the identity, supported in V, such that $h(F^N(x_0)) = x_0$. We set $\Phi = h \circ F$. It is evident that Φ coincides with F outside $\overline{B_{\epsilon_0}}$, that $d(F(y), \Phi(y)) < \delta$ and

$$\Phi^N(x_0) = (h \circ F)(F^{N-1}(x_0)) = x_0.$$

By Brouwer's translation arc theorem (Theorem A.2.3), Φ has a fixed point and we have reached a contradiction □

Lemma 5.2.2. *Q is finite.*

Proof. We may suppose that S_0 is an ideal hyperbolic polygon, with a finite number of ideal points (since S is a compact surface minus a finite number of points). It suffices to show that there are neighborhoods in S_0 of each of these ideal points whose images intersect a finite number of translates σS_0. Once this is done, the complement of these neighborhoods in S_0 is compact and the desired property will follow.

Let then *tildey* be one of the ideal points. It corresponds to a fixed point y that was removed from M. Now, let V be a topological disk around y.

The essential point of the proof is the way this neighborhood rotates around y. Note that if the image of $V \cap S_0$ by F were to intersect infinitely many translates σS_0, a radius of V would be mapped to a curve that spirals "wildly" around y. More precisely, this curve would contain segments with arbitrarily large winding number. Since f is differentiable at y, taking V to be sufficiently small (so that the points in V rotate around y acording to the differential of f at y) assures us that this does not happen. The proof of the lemma is complete. □

Now, using the subadditive ergodic theorem (Theorem A.1.3), we show that our walk in Γ has linear growth. However, we note that, a priori, this linear growth may be zero.

Lemma 5.2.3. *There exists a constant $m \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} |g(F^n(\tilde{x}))| = m$$

for μ-almost every point x in S. ($|\cdot|$ is a fixed word norm for Γ)

Proof. Set $\Omega = Q^{\mathbb{N}}$ and denote by $X_i : \Omega \to Q$ the canonical projections. Define the measurable maps

$$W_n = |X_1 \cdot \ldots \cdot X_n|.$$

W_1 is integrable:

$$\int_\Omega W_1 \, d\mathbb{P} = \sum_{\sigma \in Q} |\sigma| \cdot \mu(\sigma) < \infty.$$

The triangular inequality yields

$$W_{n+k} = |X_1 \cdot \ldots \cdot X_{n+k}| \leq |X_1 \cdot \ldots \cdot X_n| + |X_{n+1} \cdot \ldots \cdot X_{n+k}| = W_n + W_k \circ T^n.$$

Let W be the limit of $\frac{1}{n} W_n$, as in Theorem A.1.3. Since W is T-invariant and T is ergodic, W must be constant $\mathbb{P}$-a.e.. □

5.3 Positive linear growth

In this section we will prove that the linear growth just established is actually strictly positive, that is, that $m > 0$. We choose a basis $\mathcal{G}$ for the free group Γ. With respect to $\mathcal{G}$, the Cayley graph of Γ is a tree and every element of Γ admits an unique expression as a word in $\mathcal{G}$. From now on, $|\cdot|$ will denote the word norm with respect to this generating set. By simple replacement, we rewrite the sequences $w(x) = w_1 w_2 \ldots$ (w_i in Q) as $w^{\mathcal{G}}(x) = w_1^{\mathcal{G}} w_2^{\mathcal{G}} \ldots$ ($w_i^{\mathcal{G}}$ in

$\mathcal{G}$). We will denote by E the set of points x in S such that $w^{\mathcal{G}}(x)$ contains arbitrarily long trivial subwords. Since E is f-invariant, its μ-measure is either 0 or 1. We shall examine both cases separately and show that $\mu(E) = 0$ implies $m > 0$ and that $\mu(E) = 1$ is not possible. But first, we remark that the rotation vector of f gives us a sufficient condition for the positivity of m.

Lemma 5.3.1. *If the rotation vector of f is not zero, then $m > 0$.*

Proof. Since the rotation vector is not zero, there exists an element γ from the basis $\mathcal{G}$ such that the frequencies of γ and γ^{-1} in the sequence $w^{\mathcal{G}}(x)$ of a μ-generic point x are different. Since Γ is free, even after reducing $w^{\mathcal{G}}(x)$ by removing the trivial subwords, either γ or γ^{-1} remain with positive frequency. This yields $m > 0$, again because of the freeness of Γ. □

5.3.1
Case 1: $\mu(E) = 0$

Lemma 5.3.2. *If $\mu(E) = 0$, then $m > 0$.*

Proof. We choose N sufficiently large such that

$$\{x \in S \ : \ w^{\mathcal{G}}(x) \text{ contains trivial subwords of length at most } N\}$$

has positive measure and then we choose x in this set. We define $\tau(n) = w_1^{\mathcal{G}} \cdot \ldots \cdot w_n^{\mathcal{G}}(x)$. It follows that the smallest integer n_1 such that $|\tau(n)| \geq 1$ for every $n \geq n_1$ exists and satisfies $n_1 \leq N + 1$. Suppose by induction that the smallest integer n_j such that $|\tau(n)| \geq j$ for every $n \geq n_j$ exists and satisfies $n_j \leq j(N+1)$. Now we consider $\tau(n_j + k)$, for $k \geq 0$. By assumption, $\tau(n_j + N + 1) \neq \tau(n_j)$.

We claim that $|\tau(n_j + N + 1)| \geq j + 1$. Assume for a contradiction that $|\tau(n_j + N + 1)| = j$. Since the Cayley graph of Γ with respect to $\mathcal{G}$ is a tree, this assumption would imply the existence of an index i, with $0 < i < N + 1$ such that $|\tau(n_j + i)| = j - 1$. This contradicts the definition of n_j and thus proves the claim.

Furthermore, for $k \geq N + 1$, $|\tau(n_j + k)| \geq j + 1$, because $|\tau(n_j + k)| = j$ would imply that $\tau(n_j + i) = \tau(n_j)$ for an index $i > N + 1$ (again thanks to the tree structure of the Cayley graph). Finally, this also leads to a contradiction because it implies that $w^{\mathcal{G}}(x)$ contains a trivial subword longer than N. We conclude then that the smallest integer n_{j+1} such that $|\tau(n)| \geq j + 1$ for every

$n \geq n_{j+1}$ exists and satisfies $n_{j+1} \leq (j+1)(N+1)$. Taken together, all the preceding provides

$$\lim_{n\to\infty} \frac{1}{n}|\tau(n)| = \lim_{j\to\infty} \frac{1}{n_j}|\tau(n_j)| \geq \frac{j}{j(N+1)} > 0.$$

However, if we set $m_i = \sum_{l=1}^{i} |w_l|$, we also have

$$\lim_{n\to\infty} \frac{1}{n}|\tau(n)| = \lim_{i\to\infty} \frac{1}{m_i}|\tau(m_i)| = \lim_{i\to\infty} \frac{1}{m_i}|w_1 \cdot \ldots \cdot w_i| = \left(\lim_{i\to\infty} \frac{i}{m_i}\right)\left(\lim_{i\to\infty} \frac{1}{i}|w_1 \cdot \ldots \cdot w_i|\right),$$

where

$$\lim_{i\to\infty} \frac{m_i}{i} = \sum_{\sigma\in Q} \mu(\sigma) \cdot |\sigma| > 0.$$

□

5.3.2
Case 2: $\mu(E) = 1$

Lemma 5.3.3. *If $\mu(E) = 1$ then the ω-limit of almost every point in S intersects* $\mathrm{Fix}(f)$*. In other words, the orbit of almost every point in S is not precompact (in S).*

Proof. Suppose, for a contradition, that x is in E and that its orbit stays away from $\mathrm{Fix}(f)$. It follows that the orbit stays within a compact region K of S (the corresponding image of K in S_0 will still be denoted by K). By assumption, $w^{\mathcal{G}}(x)$ has arbitrarily long blocks that are equal to the identity in Γ. This means that we can find integers $n_k > m_k$ such that $n_k - m_k \to \infty$ and $g(F^{m_k}(x_0))$ and $g(F^{n_k}(x_0))$ are at a distance bounded by $2L$, where $L = \max_{\sigma\in Q} |\sigma|$. Next let

$$K^{2L} = \overline{\bigcup_{|\sigma|<2L} \sigma K}.$$

We note that K^{2L} is still compact. By suitably translating the $F^{m_k}(x)$ and extracting subsequences, we construct a sequence x_k satisfying the following conditions:

1. x_k lies in K for every k in $\mathbb{N}$;
2. for each k there exists l_k such that $y_k = F^{l_k}(x_k)$ is in K^{2L};
3. l_k is strictly increasing;
4. Both x_k and y_k are convergent.

Restricted to K^{2L}, the hyperbolic distance $d(F(p), p)$ is bounded from below by $\delta > 0$. We shall construct a diffeomorphism Φ that coincides with F outside K^{2L} and such that $d(\Phi(p), F(p)) < \delta/2$. Note that this diffeomorphism cannot have fixed points and this is the fact that will give the final contradition.

Choose positive integers M and N, with $M < N$, sufficiently large so that $d(x_M, x_N) < \delta/4$ and $d(F^{-1}(y_M), F^{-1}(y_N)) < \delta/4$. Let U be an open set of diameter smaller than $\delta/2$, containing x_M and x_N but not containing $F^j(x_N)$ for $0 \leq j \leq l_N - 1$. Analogously, let V be a neighborhood of $F^{-1}(y_M)$ and $F^{-1}(y_N)$ of diameter smaller than $\delta/2$, not containing $f^j(x_M)$ for $0 \leq j \leq l_M - 2$. Let h be a diffeomorphism isotopic to the identity, supported on $U \cup V$ and such that $h(x_M) = x_N$ and $h(F^{-1}(y_M)) = F^{-1}(y_N)$. Let $\Phi = F \circ h$. It follows that

$$\Phi^{l_N}(x_M) = \Phi^{l_N - 1}(F(x_N)) = y_N$$

and

$$\Phi^{-l_M}(y_N) = \Phi^{-l_M+1}(h^{-1}(F^{-1}(y_N))) = \Phi^{-l_M+1}(F^{-1}(y_M)) = x_M.$$

Hence, $\Phi^{l_N - l_M}(x_M) = x_M$, with $l_N - l_M > 0$. Since Φ is isotopic to the identity and $\mathbb{D}$ homeomorphic to the plane, Brouwer's translation arc theorem ensures Φ has a fixed point. $\square$

Corollary 5.3.4. $m > 0$

Proof. Since $\mathrm{supp}(\mu)$ is compact, invariant and of full measure, the orbit of almost every point remains within $\mathrm{supp}(\mu)$ and is, hence, precompact. This shows that Case 2 is not possible. $\square$

Corollary 5.3.5. *If the rotation vector of f vanishes, then the subgroup $\langle Q \rangle$ generated by Q in Γ is free of rank 2 or higher.*

Proof. $\langle Q \rangle$ is free because it is a subgroup of a free group. If its rank were 1 or 0, it would be abelian and the vanishing rotation vector would imply $m = 0$. $\square$