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A
Appendix

To say that the categories $Hom_{Cat}(\perp, X)$ and $\mathbf{1}$, the terminal category, are equivalent is the same as to say that there exists an arrow from $\perp$ to X and, for every pair of arrows $f, g\colon \perp \to X$, there exists a unique arrow from f to g which is an isomorphism.

Proof. Let us first prove the definition from the equivalence: Let $\mathbf{F}\colon Hom_{Cat}(\perp, X) \to \mathbf{1}$ be the functor that takes every object (1-cell) $\perp \to X$ to the unique $\mathbf{1}$-object $*$ and every morphism (2-cell) to the identity arrow $* \to *$ and let $\mathbf{G}\colon \mathbf{1} \to Hom_{Cat}(\perp, X)$ be the functor that takes the $\mathbf{1}$-object to an object of $Hom_{Cat}(\perp, X)$, let us say h, and the identity arrow to $id_h\colon h \to h$. We show that $id_{Hom_{Cat}} \cong \mathbf{G} \circ \mathbf{F}$.

$Hom_{Cat}(\perp, X)$ cannot be empty, otherwise there would not even exist a bijection between it and $\mathbf{1}$. Let $\tau\colon Hom_{Cat}(\perp, X) \to Hom_{Cat}(\perp, X)$ be a morphism that takes every object to h and every morphism to id_h and σ its inverse. Then

$$
\begin{tikzcd}
\perp \xrightarrow{f} X \arrow[r, shift left, "\tau_f"] \arrow[d, Rightarrow, "\alpha"] & \perp \xrightarrow{h} X \arrow[l, shift left, "\sigma_f"] \arrow[d, Rightarrow, "\mathbf{FG}(\alpha)=id_h"] \\
\perp \xrightarrow{g} X \arrow[r, shift left, "\tau_g"] & \perp \xrightarrow{h} X \arrow[l, shift left, "\sigma_g"]
\end{tikzcd}
\tag{A-1}
$$

Given a pair f, g of objects, there exists a morphism from f to g, viz., $\sigma_g \cdot id_h \cdot \tau_f$. We show now that this morphismis unique: suppose that there exists two morphisms (α and β) from f to g. As the above diagram commutes for every arrow from f to g, we have $id_h \circ \tau_f = \tau_g \circ \alpha$ and $id_h \circ \tau_f = \tau_g \circ \beta$, i.e., $\tau_g \circ \alpha = \tau_g \circ \beta$ and, as τ_g is an isomorphism, $\alpha = \beta$.

Now we show the equivalence form the definition: As $Hom_{Cat}(\perp, X)$ has at least one object, we can define functors $\mathbf{F}$ and $\mathbf{G}$ as in the first part of this proof. As there exists arrows between every pair of objects, we can define a diagram like (A-1) and the arrows being isomorphisms guarantee to us that every arrow from f to g corresponds to a unique arrow from h to h and conversely.

$\square$

B
Appendix

To say that the categories $Hom_{Cat}(X, A \times B)$ and $Hom_{Cat}(X, A) \otimes Hom_{Cat}(X, B)$ are equivalent is the same as to say that, for any $f\colon X \to A$ and $g\colon X \to B$, there exist $h\colon X \to A \times B$ and isomorphisms $\pi_1 \circ h \cong f$ and $\pi_1 \circ h \cong g$ such that, for all $k\colon X \to A \times B$ and 2-cells $\alpha\colon \pi_1 \circ h \Rightarrow \pi_1 \circ k$ and $\beta\colon \pi_2 \circ h \Rightarrow \pi_2 \circ k$, there exist a unique $\gamma\colon h \Rightarrow k$ such that $id_{\pi_1}; \gamma = \alpha$ and $id_{\pi_2}; \gamma = \beta$.

Proof. First, we prove that the definition of 2-product comes from the equivalence of the aforesaid categories:

As the categories are equivalent, there are functors $\mathbf{F}\colon Hom_{Cat}(X, A \times B) \to Hom_{Cat}(X, A) \otimes Hom_{Cat}(X, B)$ and $\mathbf{G}\colon Hom_{Cat}(X, A) \otimes Hom_{Cat}(X, B) \to Hom_{Cat}(X, A \times B)$ such that $id_\times \cong \mathbf{G} \circ \mathbf{F}$ and $id_\otimes \cong \mathbf{F} \circ \mathbf{G}$.

Let us define $\mathbf{F}$ as the functor that takes h to $(\pi_1 \circ h, \pi_2 \circ h)$ and γ to $(id_{\pi_1}; \gamma, id_{\pi_2}; \gamma)$ for every object (1-cell) h and arrow (2-cell) γ of $Hom_{Cat}(X, A \times B)$ and let us define $\mathbf{G}$ as the functor that takes (f, g) to $\langle f, g \rangle$ and (α, β) to $\alpha \mid \beta$, for every object (f, g) and arrow (α, β) of $Hom_{Cat}(X, A) \otimes Hom_{Cat}(X, B)$.

Let f and g be objects such that $f \in Hom_{Cat}(X, A)$ and $g \in Hom_{Cat}(X, B)$. Then, $\langle f, g \rangle \in Hom_{Cat}(X, A \times B)$. Let us take $h = \langle f, g \rangle$. As $id_\otimes \cong \mathbf{F} \circ \mathbf{G}$, then $id(f, g) = (f, g) \cong F \circ \mathbf{G}(f, g) = \mathbf{F}(\langle f, g \rangle) = (\pi_1 \circ h, \pi_2 \circ h)$, i.e.,

$$f \cong \pi_1 \circ h \quad \text{(I)} \qquad \text{and} \qquad g \cong \pi_2 \circ h \quad \text{(II)}$$

Let k be an object of $Hom_{Cat}(X, A \times B)$, α be an arrow of $Hom_{Cat}(X, A)$ such that $\alpha\colon \pi_1 \circ h \Rightarrow \pi_1 \circ k$ and β be an arrow of $Hom_{Cat}(X, B)$ such that $\beta\colon \pi_2 \circ h \Rightarrow \pi_2 \circ k$.

As there is an arrow from f to $\pi_1 \circ h$ (from (I)) and $\alpha\colon \pi_1 \circ h \Rightarrow \pi_1 \circ k$, there is an arrow from f to $\pi_1 \circ k$. As there is an arrow from g to $\pi_2 \circ h$ (from (II)) and $\beta\colon \pi_2 \circ h \Rightarrow \pi_2 \circ k$, there is an arrow from g to $\pi_2 \circ k$. Thus, $\langle f, g \rangle \Rightarrow \langle \pi_1 \circ k, \pi_2 \circ k \rangle \cong k$, i.e., $h \Rightarrow k$.

From the natural isomorphisms, we have that $\mathbf{G}$ is left adjoint to $\mathbf{F}$. Then,

$$\begin{aligned} (\mathbf{G}(f,g),k) &\cong ((f,g),\mathbf{F}k) \\ (\langle f,g\rangle,k) &\cong ((f,g),(\pi_1 \circ k,\pi_2 \circ k)) \end{aligned}$$

Using (I) and (II), we have that

$$(\langle f,g\rangle,k) \cong ((\pi_1 \circ h,\pi_2 \circ h),(\pi_1 \circ k,\pi_2 \circ k))$$

As for every arrow in $(\langle f,g\rangle,k)$ there is only one arrow in $((\pi_1 \circ h,\pi_2 \circ h),(\pi_1 \circ k,\pi_2 \circ k))$ and conversely, we have that γ is unique and, as both (α,β) and $(id_{\pi_1};\gamma, id_{\pi_2};\gamma)$ are arrows in $((\pi_1 \circ h,\pi_2 \circ h),(\pi_1 \circ k,\pi_2 \circ k))$, we have that $\alpha = id_{\pi_1};\gamma$ and $\beta = id_{\pi_2};\gamma$.

Now we prove that the equivalence comes from the definition of 2-product.

According to the definition of 2-product, we can define functors $\mathbf{F}$ and $\mathbf{G}$, such that $\mathbf{G}(f,g) = h$, for every object (f,g) in $Hom_{Cat}(X,A) \otimes Hom_{Cat}(X,B)$, and $\mathbf{G}(\alpha,\beta) = \gamma$, for every arrow (α,β) in $Hom_{Cat}(X,A) \otimes Hom_{Cat}(X,B)$ and $\mathbf{F}(h) = (id_{\pi_1} \circ h, id_{\pi_2} \circ h)$, for every h in $Hom_{Cat}(X,A\times B)$, and $\mathbf{F}(\gamma) = (id_{\pi_1};\gamma, id_{\pi_2};\gamma)$, for every $\gamma \in Hom_{Cat}(X,A\times B)$.

Let us define $\tau\colon Hom_{Cat}(X,A\times B) \to Hom_{Cat}(X,A\times B)$ as the arrow that takes h to $\langle \pi_1 \circ h,\pi_2 \circ h\rangle$ and γ to $id_{\pi_1};\gamma \mid id_{\pi_2};\gamma$, for every object h and arrow γ of $Hom_{Cat}(X,A\times B)$. It is easy to see that this arrow has an inverse and that

$$\begin{tikzcd} h \arrow[r, "\tau_h"] \arrow[d, Rightarrow, "\gamma"'] & \langle \pi_1 \circ h,\pi_2 \circ h\rangle \arrow[d, Rightarrow, "id_{\pi_1};\gamma \mid id_{\pi_2};\gamma"] \\ k \arrow[r, "\tau_k"'] & \langle \pi_1 \circ k,\pi_2 \circ k\rangle \end{tikzcd}$$

commutes. So, there is a natural isomorphism $id_\times \cong \mathbf{G}\circ\mathbf{F}$.

Let us define $\sigma\colon Hom_{Cat}(X,A) \otimes Hom_{Cat}(X,B) \to Hom_{Cat}(X,A) \otimes Hom_{Cat}(X,B)$ as the arrow that takes (f,g) to $(\pi_1\circ\langle f,g\rangle, \pi_1\circ\langle f,g\rangle)$ and (α,β) to $(id_{\pi_1};(\alpha \mid \beta), id_{\pi_2};(\alpha \mid \beta))$, for every object f and arrow α of $Hom_{Cat}(X,A)$ and every object g and arrow β of $Hom_{Cat}(X,B)$. It is easy to see that this arrow has an inverse and that

$$\begin{tikzcd}
(f,g) \arrow[r, "\sigma_{(f,g)}"] \arrow[d, Rightarrow, "(\alpha,\beta)"'] & (\pi_1 \circ h, \pi_2 \circ h) \arrow[d, Rightarrow, "(id_{\pi_1};\gamma, id_{\pi_2};\gamma)"] \\
(f',g') \arrow[r, "\sigma_{(f',g')}"'] & (\pi_1 \circ k, \pi_2 \circ k)
\end{tikzcd}$$

commutes. So, there is a natural isomorphism $id_{\otimes} \cong \mathbf{F} \circ \mathbf{G}$.

□