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Asymptotic Behavior of Systems with Two-Input Compensators

PEDRO M. G. FERREIRA

Abstract—**This note completely solves the robust asymptotic tracking-disturbance rejection problem for feedback systems with two-input, one-output plant and compensator.**

I. Introduction

Consider the following general two-input, one-output plant and compensator of Fig. 1.

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The author is with the Department of Electrical Engineering, Pontifica Universidade Católica, Rio de Janeiro, Brazil.

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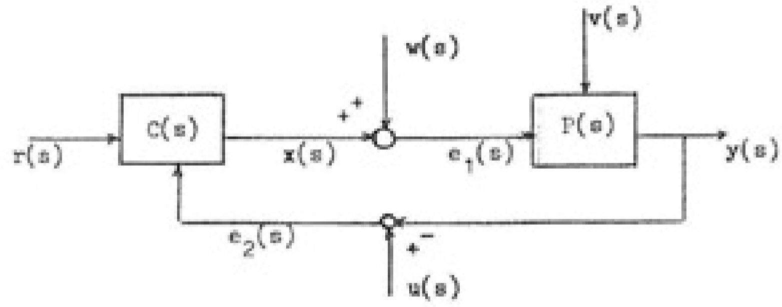

Fig. 1.

The plant to be controlled, $P(s)$, is a proper transfer function matrix; $r(s)$ is a q-valued vector representing the Laplace transform of a signal to be asymptotically tracked at the plant's output. $u(s)$, $w(s)$, and $v(s)$ are Laplace transforms of signals to be asymptotically rejected at the plant's output. The dimensions of these vectors are q, m, and n, respectively. The four rational vectors are assumed to be proper.

The classes of exogenous signals are assumed to be known, i.e., we know the denominator of each entry of $r(s)$, $u(s)$, $w(s)$, and $v(s)$. The least common denominators of these entries will be denoted by $\alpha_r(s)$, $\alpha_u(s)$, $\alpha_w(s)$, and $\alpha_v(s)$, respectively.

All factorizations in this note are over the ring of proper stable rational functions, denoted by S, a Euclidean domain. As a consequence, α_r, α_u, α_w, and α_v are stable and proper rational functions, in fact, biproper. It is clear that r, w, u, and v can be expressed in the following way:

$$r=\tilde{D}_r^{-1}r_o,\ w=\tilde{D}_w^{-1}w_o,\ u=\tilde{D}_u^{-1}u_o,\ v=\tilde{D}_v^{-1}v_o, \text{ where } \tilde{D}_r, \tilde{D}_w, \tilde{D}_u,$$

$\tilde{D}_v \in M(S)$, i.e., they are proper and stable rational matrices. In fact, they are biproper, r_o, w_o, v_o, and u_o are proper and stable rational vectors, i.e., also belong to $M(S)$ and are related to the (arbitrary) initial conditions of the exogenous signals.

$\tilde{D}_r$, $\tilde{D}_w$, $\tilde{D}_u$, and $\tilde{D}_v$ may be assumed diagonal matrices for all practical purposes. It is clear that α_r, α_u, α_w, and α_v are the largest invariant factors of $\tilde{D}_r$, $\tilde{D}_u$, $\tilde{D}_w$, and $\tilde{D}_v$, respectively.

Let $P = \tilde{D}^{-1}[\tilde{N}_1|\tilde{N}]$ be a left coprime (l.c.) factorization of the transfer function matrix between y and $[^{v}_{e_1}]$. Let $C = \tilde{D}_c^{-1}[\tilde{N}_{c1}|\tilde{N}_{c2}]$ be a l.c. factorization of the transfer function matrix between x and $[^{r}_{e_2}]$. It is well known [1], [3] that loop stability implies $(\tilde{D}, \tilde{N})$ l.c. as well as $(\tilde{D}_c, \tilde{N}_{c2})$ l.c. Define the right coprime (r.c.) factorizations $ND^{-1} = \tilde{D}^{-1}\tilde{N}$ and $N_{c2}D_c^{-1} = \tilde{D}_c^{-1}\tilde{N}_{c2}$. It is supposed that the plant can have its parameters perturbed, i.e., $N \leftarrow N^*$, $D \leftarrow D^*$, etc., the perturbations being defined as in [2].

It is well known that when the loop is stable, the above factorizations can be chosen such that

$$\begin{bmatrix} \tilde{D}_c & \tilde{N}_{c2} \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -N_{c2} \\ N & D_c \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{1}$$

Define:

$$M(N^*, D^*) := \tilde{N}_{c2}N^* + \tilde{D}_cD^*. \tag{2}$$

It is clear from (1) that if the loop is stable, then $M(N, D) = I$.

Our problem can be formulated this way: given P and the classes of exogenous signals, find the conditions for solvability and solve for C such that loop is stable, r is asymptotically tracked, and w, u, and v are asymptotically rejected, even if P is subject to arbitrary but "small" perturbations. (The meaning of "small" perturbations is given in [2].)

The asymptotic tracking/disturbance rejection problem has been often studied in the past 15 years. Referring to some of the most important papers in the past few years, [4] studied the problem with two-output plants. This same problem was studied by [5] and [6] in a more general context, namely, that of synthesizing a transfer function through feedback compensation. Reference [6] studies the problem in both transfer function and state-space settings and achieves a deep understanding of the structure in terms of signal flow. Most recently [7] introduced the concept of "containment" to determine the achievable transfer functions in a feedback loop, while in [1] this same problem is solved using the factorization approach.

Papers [1] and [4]–[7] handle the two-output plant problem, while the present note tackles the one-output plant problem, which is simpler. On the other hand, the present note addresses the robustness issue more explicitly than those papers.

The solution of the two-input compensator, one-output plant robust asymptotic tracking problem is given in [2], while the asymptotic rejection of v, w, and u in Fig. 1 turns out to be a one-input compensator, one-output plant problem.

However, [2] does not prove the conditions for the solvability of the above tracking problem. This will be done here. Besides, to the author's knowledge, the solution of the asymptotic rejection of the three disturbances is not available in the literature in the present setup. It will be given here for the sake of completeness, in the vein of [2].

II. The Tracking Problem

Vidyasagar [2, pp. 304 ff.] proved with skill and mathematical rigor, assuming $q \leqslant m$ and disjointness of the zeros of N from those of α_r, that C robustly solves the asymptotic tracking (with loop stability) problem if and only if:

i) $M(N^*, D^*)$ is unimodular $\forall(N^*, D^*)$ in some neighborhood of (N, D), (3)

ii) $\alpha_r^{-1}D_c \in M(S)$, (4)

iii) $N^*[M(N^*, D^*)]^{-1}(\tilde{N}_{c2}-\tilde{N}_{c1})\tilde{D}_r^{-1} \in M(S)$. (5)

Moreover, if N is square, iii) is equivalent to

iii') $(\tilde{N}_{c2}-\tilde{N}_{c1})\tilde{D}_r^{-1} \in M(S)$. (6)

Remark: The disjointness of the zeros of N from those of α_r as well as condition i) above were omitted in the formulation of the theorem by [2]. They were used, however, in the proof of the theorem.

The main contribution of this paper is as follows.

Lemma 1: The robust asymptotic tracking problem is solvable if and only if $q \leqslant m$ and the zeros of N are disjoint from those of α_r.

Proof:

(Only If): First notice that $\tilde{N}_{c1}$ and $\tilde{D}_r$ have to be r.c. If this were not the case, some mode of r would be cancelled by $\tilde{N}_{c1}$ and so could not appear at y, since a stable loop does not generate any unstable mode. (A formal proof of the right coprimeness of $\tilde{N}_{c1}$ and $\tilde{D}_r$ is given in [2].) As a consequence, if some nontrivial factor divides every element of $\tilde{D}_r$, or in other words, if no invariant factor of $\tilde{D}_r$ is 1, we have $q \leqslant m$, since $\tilde{N}_{c1}$ is a $m \times q$ matrix.

However, if the smallest invariant factor of $\tilde{D}_r$ is 1, a different proof has to be given. It is easy to check that

$$y = N^*[M(N^*, D^*)]^{-1}\tilde{N}_{c1}r.$$

Hence, for the nominal plant,

$$y = N\tilde{N}_{c1}r.$$

Consider now the particular situation when all, but the first, components of r are zero. Let r_1 denote the first component of r. Let n_i^T, $i \in q$, denote the ith row of N, and let $\tilde{n}_1$ denote the first column of $\tilde{N}_{c1}$. One obtains

$$y = \begin{bmatrix} n_1^T \\ \vdots \\ n_q^T \end{bmatrix} \tilde{n}_1 r_1.$$

Now asymptotic tracking implies

$$n_1^T\tilde{n}_1r_1 = r_1 + p_1 \tag{7}$$

$$n_i^T\tilde{n}_1r_1 = p_i, \qquad i = 2, 3, \cdots, q \tag{8}$$

where p_i, $i \in q$, are proper and stable rational functions. Now suppose that $q > m$. Let $a_i \in S$ be such that

$$a_1n_1^T + a_2n_2^T + \cdots + a_{m+1}n_{m+1}^T = 0. \tag{9}$$

We may assume $n_2^T, \cdots, n_{m+1}^T$ linearly independent. If this were not the case, we would perturb slightly at the outset one of these rows. Hence, $a_1 \neq 0$. We may assume also (perturbing slightly N at the outset, if necessary) that $a_1 r_1 \notin S$, since $r_1 \notin S$. From (7) through (9)

$$a_1(r_1+p_1)+a_2p_2+\cdots+a_{m+1}p_{m+1}=0$$

which leads to a contradiction, since

$$\sum_{i=1}^{m+1} a_i p_i \in S$$

establishing the necessity of $q \leqslant m$.

With $q \leqslant m$ slightly perturb N at the outset, if necessary, so that $\alpha_r I$ and N are l.c. Then [2] proves the necessity of (4). But loop-stability implies noncancellation of unstable modes. Hence, (4) implies the left coprimeness of α_r and N for the nominal plant also, or equivalently, the disjointness of the zeros of N from those of α_r.

(If): If the conditions of the lemma are satisfied, there exists $N_{c2}\tilde{D}_c^{-1}$ which stabilizes $\alpha_r^{-1}ND^{-1}$. Let $\tilde{D}_c^{-1}\tilde{N}_{c2}$ be a l.c. factorization of $N_{c2}\tilde{D}_c^{-1}\alpha_r^{-1}$. Let $\tilde{N}_{c1} = \tilde{N}_{c2} - Q\tilde{D}_r$, with $Q \in M(S)$, arbitrary otherwise. Then $\tilde{D}_c^{-1}[\tilde{N}_{c1} \vdots \tilde{N}_{c2}]$ solves the problem, according to (3)–(5), concluding the proof of the lemma.

Remark: If $m = q$, a closed form of C can be obtained, using (6): see the last section of this note where the general problem is handled.

III. Disturbance Rejection

A. Asymptotic Rejection of w

The conditions of solvability are well known in this case: $q \leqslant m$ and zeros of N disjoint from those of α_w. If these conditions are satisfied, it is well known that the class of compensators $\tilde{D}_c^{-1}\tilde{N}_{c2}$ which solve the problem is given by $\{C_1\alpha_w^{-1}: C_1 \text{ stabilizes } \alpha_w^{-1}ND^{-1}\}$.

The proof of this result closely follows that of the tracking problem with a one-input compensator, and a rigorous proof of this last problem can be found in [2] and [8].

B. Asymptotic Rejection of u

This problem is less discussed in the literature and, in fact, a complete solution of it is missing (to the knowledge of the author) so it will be given here.

Lemma 2: The robust problem of asymptotically rejecting u with loop stability has a solution if and only if $\tilde{D}_u$ and $\tilde{D}$ are r.c., whatever the relationship may be between m and q. If that condition is satisfied, the class of all compensators which solve the problem is given by $\{C_1\tilde{D}_u: C_1 \text{ stabilizes } \tilde{D}_u ND^{-1}\}$.

Proof: It is easy to see that

$$y=N^*[M(N^*, D^*)]^{-1}\tilde{N}_{c2}\tilde{D}_u^{-1}u_o.$$

Since $u_o \in M(S)$ is arbitrary, robust asymptotic rejection of u is equivalent to

$$N^*[M(N^*, D^*)]^{-1}\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S) \ \forall (N^*, D^*) \text{ in some neighborhood of } (N, D). \tag{10}$$

(Only if): The above implies, in view of (1) and (2),

$$N\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S). \tag{10a}$$

First suppose $q \geqslant m$.

Perturb N slightly at the outset, if necessary, so that its zeros are disjoint from those of $\tilde{D}_u$. Let A and B be l.c. such that $A^{-1}B = \tilde{N}_{c2}\tilde{D}_u^{-1}$. Hence, $NA^{-1}B \in M(S)$. But N and A are r.c. because its zeros are disjoint. As a consequence, A is unimodular, and hence $\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S)$. This and loop stability imply right coprimeness of $\tilde{D}_u$ and $\tilde{D}$.

Suppose now $q < m$. Then,

$$M(N, D^*)=I+\tilde{D}_c(D^*-D).$$

It can be shown [2, p. 307] that $[M(N, D^*)]^{-1} = I + \epsilon\tilde{D}_c Q$, with $\epsilon \in \mathfrak{R}$, sufficiently small in absolute value and $Q \in M(S)$, otherwise arbitrary. Then from (10),

$$N\tilde{N}_{c2}\tilde{D}_u^{-1}+\epsilon N\tilde{D}_c Q\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S) \quad \forall Q \in M(S)$$

which implies, in view of (10a),

$$N\tilde{D}_c Q\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S) \quad \forall Q \in M(S). \tag{11}$$

Now it is easy to see that the problem is solved with (N^*, D^*) and $(\tilde{D}_c, \tilde{N}_{c2})$ if and only if it is solved with $(N^*, V^{-1}D^*)$ and $(\tilde{D}_c V, \tilde{N}_{c2})$ $\forall V$ unimodular.

Next notice that

$$V^{-1}\tilde{D}_c^{-1}\tilde{N}_{c2}=V^{-1}\tilde{D}_c^{-1}U^{-1}U\tilde{N}_{c2}$$

with $U\tilde{D}_c V$ and $U\tilde{N}_{c2}$ l.c. $\forall U, V$ unimodulars.

Choose U and V unimodulars such that $U\tilde{D}_c V$ is in Smith form. Redefine $D^*(\text{new}) = V^{-1}D^*(\text{old})$; $\tilde{D}_c(\text{new}) = U\tilde{D}_c(\text{old})V$. Then

$$\tilde{D}_c =: \text{diag}\,(\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_m); \ \tilde{d}_i^{-1}\tilde{d}_{i+1} \in S.$$

Now $m > q$ and left coprimeness of $\tilde{D}_c$ and $\tilde{N}_{c2}$ imply $\tilde{d}_1 = 1$. Let n_i, $i \in \mathbf{m}$, denote the ith column of N and let the ith row of $\tilde{N}_{c2}$ be denoted by $[\tilde{n}_{i1}, \tilde{n}_{i2}, \cdots, \tilde{n}_{iq}]$. Let q_{ij}, $i, j \in \mathbf{m}$, denote the entries of Q and let k be any fixed integer $1 \leqslant k \leqslant m$. Choose Q such that $q_{1k} = 1$, $q_{ij} = 0$ $\forall (i, j) \neq (1, k)$. Then it is easy to check that

$$N\tilde{D}_c Q\tilde{N}_{c2}=n_1[\tilde{n}_{k1}, \tilde{n}_{k2}, \cdots, \tilde{n}_{kq}]. \tag{12}$$

Let d_i, $i \in \mathbf{q}$ be the denominators of the entries of u, i.e., $\tilde{D}_u = \text{diag}\,(d_1, d_2, \cdots, d_q)$. It can be assumed that $n_1 d_i^{-1} \notin M(S)$. (If this were not the case, N would be perturbed at the outset.) From this, (11), and (12) we have $\tilde{n}_{ki}d_i^{-1} \in S$, $i \in \mathbf{q}$. Now let $k = 1, 2, \cdots, m$ to get $\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S)$. From this and loop stability one concludes the right coprimeness of $\tilde{D}_u$ $\tilde{D}$.

(If): Let $\tilde{D}_c^{-1}\tilde{N}_{c2} = \tilde{D}_c^{-1}\tilde{N}_{c2}\tilde{D}_u$, with $\tilde{D}_c^{-1}\tilde{N}_{c2}$ stabilizing $\tilde{D}_u\tilde{D}^{-1}\tilde{N}$. From (10) it is seen that this compensator robustly solves the problem.

C. Asymptotic Rejection of v

Like the preceding one, this problem has not had careful attention in the literature. Its solution is structurally equal to the solution of the first rejection problem, namely, $q \leqslant m$ and zeros of N disjoint from those of $\tilde{D}_v$, the compensator which solves the problem $C_1\alpha_v^{-1}$, with C_1 stabilizing $\alpha_v^{-1}ND^{-1}$. The proof of this result is much simpler though, because to prove the necessity of the conditions, it is enough to perturb $\tilde{N}_1$, which is outside the loop and, by the way, can have "large" (not only "small") perturbations. Indeed, $y = D_c\tilde{N}_1^*\tilde{D}_v^{-1}v_o$, where $\tilde{N}_1^*$ is the perturbed $\tilde{N}_1$. Hence, $D_c\tilde{N}_1^*\tilde{D}_v^{-1} \in M(S)$, and this implies $D_c(\tilde{N}_1^* - \tilde{N}_1)\tilde{D}_v^{-1} \in M(S)$. Therefore,

$$D_c(\tilde{N}_1^*-\tilde{N}_1)UU^{-1}\tilde{D}_v^{-1}V^{-1} \in M(S) \quad \forall U, V \text{ unimodulars}. \tag{13}$$

Choose U and V such that $V\tilde{D}_v U$ is in the Smith form and choose $\tilde{N}_1^*$ such that $(\tilde{N}_1^* - \tilde{N}_1)U$ has all its elements equal to zero, but the (q, n)th one, which is chosen equal to 1. Then it can be checked that (13) implies that α_v has to be a factor of all the entries of the last column of D_c. Next choose $\tilde{N}_1^*$ such that $(\tilde{N}_1^* - \tilde{N}_1)U$ has all its elements equal to zero, but the $(q-1, n)$th one, which is chosen equal to 1, to conclude that α_v has to divide all the elements of the $(q-1)$th column of D_c, and so on.

IV. Conclusions

The following theorem summarizes the results of this note. It has not appeared elsewhere, to the knowledge of the author.

Theorem: The general problem studied in this note has a solution only if

a) $q \leqslant m$,

b) zeros of N are disjoint from the zeros of $\tilde{D}_r$, $\tilde{D}_w$, and $\tilde{D}_v$,

c) $\tilde{D}_u$ and $\tilde{D}$ are r.c.

If the above conditions are satisfied and if:

d) zeros of $\tilde{D}_u$ are disjoint from the zeros of $\tilde{D}_r$, $\tilde{D}_w$, and $\tilde{D}_v$, then the problem has a solution and the class of compensators which solve it is given by

$$C = \tilde{D}_c^{-1}[\tilde{N}_{c1} \,\vdots\, \tilde{N}_{c2}],\ \tilde{D}_c^{-1}\tilde{N}_{c2} = N_{c2}D_c^{-1}, \text{ with}$$

i) $\tilde{D}_c^{-1}\tilde{N}_{c2}$ a stabilizer of ND^{-1},

ii) $\tilde{D}_c\alpha_r^{-1}\alpha_w^{-1}\alpha_v^{-1} \in M(S)$,

iii) $N^*[M(N^*, D^*)]^{-1}(\tilde{N}_{c2} - \tilde{N}_{c1})\tilde{D}_r^{-1} \in M(S)\ \forall (N^*, D^*)$ in some neighborhood of (N, D),

iv) $\tilde{N}_{c2}\tilde{D}_u^{-1} \in M(S)$.

To conclude this note it is well worth noticing that if $q = m$, one obtains a crisp result: an "if and only if" solvability condition and the class of compensators which solve the problem in a closed form.

Corollary: Let $q = m$. The robust general problem has a solution if and only if:

a) modes of u disjoint from those of r, w, and v,

b) zeros of N disjoint from the modes of r, w, and v,

c) $\tilde{D}_u$ and $\tilde{D}_r$ r.c.

If these conditions are satisfied, the class of compensators which solve the problem is given by

$$C = (\alpha_r\alpha_w\alpha_v)^{-1}\hat{D}_c^{-1}[\hat{N}_{c2}\tilde{D}_u - Q_1\tilde{D}_r \,\vdots\, \hat{N}_{c2}\tilde{D}_u], \text{ with } Q_1 \in M(S)$$

arbitrary otherwise, $\hat{D}_{c2}^{-1}\hat{N}_{c2}$ is a stabilizer of $\tilde{D}_u ND^{-1}(\alpha_r\alpha_w\alpha_v)^{-1}$.

The problem is solved if the loop stabilizer is perturbed also.

Proof: The proof is straightforward and is omitted in the interests of brevity.

Remark: It is well known that a loop stabilizer has one degree of freedom, i.e., $\hat{D}_c$ and $\hat{N}_{c2}$ are affinite functions of an arbitrary $Q_2 \in M(S)$. After solving the robust tracking and threefold disturbance rejection problem, we are still left with a compensator with two free parameters, namely Q_1 and Q_2, both belonging to $M(S)$.

References

[1] C. A. Desoer and A. N. Güness, "Algebraic theory of linear time-invariant feedback systems with two-input two-output plant and compensator," Electron. Res. Lab., Univ. Calif., Berkeley, CA, Memo. UCB-ERL M87-1, Jan. 20, 1987; also to appear in *Intern. J. Contr.*

[2] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: M.I.T. Press, 1985.

[3] S. P. Bhattacharyya and J. W. Howze, "Transfer function conditions for stabilizability," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 253-255, 1984.

[4] L. Cheng and J. B. Pearson, "Synthesis of linear multivariable regulators," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 194-202, 1981.

[5] L. Pernebo, "An algebraic theory for the design of controllers for linear multivariable systems. Parts I and II," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 171-194, 1981.

[6] D. Y. Ohm, J. W. Howze, and S. P. Bhattacharyya, "Structural synthesis of multivariable controllers," *Automatica*, vol. 21, pp. 35-55, 1985.

[7] C. N. Nett, "Algebraic aspects of linear control system stability," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 941-949, 1986.

[8] B. A. Francis and M. Vidyasagar, "Algebraic and topological aspects of the regulator problem for lumped linear systems," *Automatica*, vol. 19, pp. 87-90, 1983.