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Introduction

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ focusing on minimal and constant mean curvature surfaces. This was initiated by Harold Rosenberg, see (19). Many works are devoted to studying the geometry of surfaces in homogeneous 3-manifolds.

The spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ form part of the 3-dimensional Thurston geometries. Namely, the classification of complete simply connected homogeneous 3-manifolds was established by W. Thurston. There is a complete list of these homogeneous 3-manifold.

(i) The canonical space forms $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$, whose isometry group is 6-dimensional.

(ii) When the isometry group of the 3-dimensional manifold has dimension 4, we label these spaces as $E^3(\kappa, \tau)$. Thus we have the next cases,

- $E^3(\kappa, \tau) = \mathbb{H}^2(\kappa) \times \mathbb{R}$, if $\kappa < 0$ and $\tau = 0$
- $E^3(\kappa, \tau) = \mathbb{S}^2(\kappa) \times \mathbb{R}$, if $\kappa > 0$ and $\tau = 0$
- $E^3(\kappa, \tau) = Nil_3$ (Heisenberg space) if $\kappa = 0$ and $\tau \neq 0$
- $E^3(\kappa, \tau) = \widetilde{PSL}_2(\mathbb{R}, \tau)$, if $\kappa < 0$ and $\tau \neq 0$

An important manifold which does not appears in this list is the so-called sphere of Berger. The spheres of Berger $E^3(\kappa, \tau) = \mathbb{S}^3_\tau$ are missing there, since their isometry group is contained in the isometry group of $\mathbb{S}^3$.

(iii) When the dimension of the isometry group is 3, the manifold has the geometry of the Lie group $Sol3$.

In a recent paper José Espinar and Harold Rosenberg proved that if Σ is a complete immersed H surface (surface having constant mean curvature H) in $\mathbb{H}^2 \times \mathbb{R}$ whose angle function does not change sign and $H > 1/2$, then Σ is a vertical cylinder over a complete curve $\gamma \subset \mathbb{H}^2$ of constant geodesic curvature $2H$, see (8, Theorem 4.1). They were inspired by the ideas of the paper "On complete mean curvature $1/2$ surfaces in $\mathbb{H}^2\times\mathbb{R}$" written by Laurent Hauswirth, Harold Rosenberg and Joel Spruck. In that paper they proved that, if Σ is a complete immersed surface in $\mathbb{H}^2\times\mathbb{R}$ having constant mean curvature $H=1/2$ and if Σ is transverse to the vertical field E_3, then Σ is an entire vertical graph over $\mathbb{H}^2$, see (5, Theorem 1.2). We use similar ideas to extend the work of Jose Espinar and Harold Rosenberg to the spaces $E^3(\kappa,\tau)$, see Theorem 4.4.1.

On the other hand, we also study the surfaces immersed in $E^3(\kappa,\tau)$ having constant mean curvature which are invariant by one-parameter group of isometries.

On (16) Ricardo Sa Earp and Eric Toubiana studied the screw motions surfaces immersed in $\mathbb{H}^2\times\mathbb{R}$ having constant mean curvature. In particular, they gave explicit formulas to screw motion surfaces. On (18) Barbara Nelli, Ricardo Sa Earp, Walcy Santos and Eric Toubiana complete the study of the geometric behavior of rotational surfaces immersed in $\mathbb{H}^2\times\mathbb{R}$.

On (3), Ricardo Sa Earp completes the study of parabolic and hyperbolic screw motions surfaces immersed in $\mathbb{H}^2\times\mathbb{R}$. There, he also gave explicit formulas and several examples.

To study this kind of surfaces, the authors have solved a second order differential equation (ODE). They found the first integral for this ODE. On (11, Apendix A), Magdalena Rodrigues, Laurent Mazet and Harold Rosenberg gave an alternative form to find the first integral for surfaces having constant mean curvature which are invariant by one-parameter group of isometries immersed in $\mathbb{H}^2\times\mathbb{R}$.

We follow the ideas presented on (16), (3) and (11) to study the surfaces having constant mean curvature which are invariant by one-parameter group of isometries immersed in $\widetilde{PSL}_2(\mathbb{R},\tau)$. In particular we give explicit formulas for rotational surfaces, surfaces invariant by parabolic isometries as well as for surfaces invariant by hyperbolic isometries.

In the case of rotational surfaces, we follow the ideas of Ricardo Sa Earp and Barbara Nelli and extend the so-called half-space theorem for $\frac{1}{2}$ rotational surfaces, to the space $\widetilde{PSL}_2(\mathbb{R},\tau)$, see (14).