

## 5

### The space $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$

In this chapter we focus our attention in the study of the geometry of the space  $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$ . We wish to study the  $H$ -surfaces, which are invariant by one-parameter group of isometries.

It will be made by understanding the isometries of  $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$  as well as the relationship between the space  $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$  and the hyperbolic space.

The space  $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$  is a simply connected homogeneous manifold whose isometry groups has dimension 4, such a manifold is a Riemannian submersion over the 2-dimensional hyperbolic space, which we will denote by  $M^2 = M^2(-1)$ . Thus, we only consider the case when the Gaussian curvature is equal to  $-1$ , that is,  $\kappa = -1$ .

Throughout this chapter, we will follow the ideas of Eric Toubiana, see (22). Recall that, we have,

$$\pi : \widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau) \longrightarrow M^2$$

the Riemannian submersion; for  $p \in M^2$ , the fibers  $\pi^{-1}(p)$  are geodesics and there exists a one-parameter family of translations along the fibers, generated by the unit Killing field  $E_3$ .

### 5.1

#### Isometries of $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$

Since there exist a Riemannian submersion  $\pi : \widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau) \longrightarrow M^2$ , the isometries of  $\widetilde{\mathrm{PSL}}_2(\mathbb{R}, \tau)$  are strongly related with the isometries of the hyperbolic space  $M^2$ .

From now on we identify the Euclidean space  $\mathbb{R}^2$  with the set of complex numbers  $\mathbb{C}$ , more precisely  $z = x + iy \approx (x, y)$ .

So, if we take  $M^2 \equiv \mathbb{D}^2$ , we obtain:

$$\widetilde{\mathrm{PSL}}_2(\mathbb{R}) = \{(z, t) \in \mathbb{R}^3; x^2 + y^2 < 1, t \in \mathbb{R}\}$$

endowed with metric,

$$d\sigma^2 = \lambda^2(z)|dz|^2 + (i\tau\lambda(\bar{z}dz - zd\bar{z}) + dt)^2$$

If we take  $M^2 \equiv \mathbb{H}^2$ , we obtain:

$$\widetilde{PSL}_2(\mathbb{R}) = \{(z, t) \in \mathbb{R}^3; y > 0, t \in \mathbb{R}\}$$

endowed with metric,

$$d\sigma^2 = \lambda^2(z)|dz|^2 + (-\tau\lambda(dz + d\bar{z}) + dt)^2$$

**Remark 5.1.1.** Let  $F$  an isometry of  $\widetilde{PSL}_2(\mathbb{R}, \tau)$ . As

$$\pi : \widetilde{PSL}_2(\mathbb{R}, \tau) \longrightarrow M^2$$

is a Riemannian submersion, we can write  $F$  in the form  $F(z, t) = (f(z), h(z, t))$ , where

$$f : M^2 \longrightarrow M^2$$

is an isometry of the hyperbolic space  $M^2$ .

To see this, we take  $p \in M^2$  and  $u, v \in T_p M^2$ . Denotes by  $\bar{u}, \bar{v}$  the horizontal lifts of  $u$  and  $v$  respectively. Let  $\bar{p}$  be the point over the fiber  $\pi^{-1}(p)$  such that  $\bar{u}, \bar{v} \in T_{\bar{p}} \widetilde{PSL}_2(\mathbb{R}, \tau)$ .

Denoting by  $dF(\bar{u}) = \bar{a}$  and  $dF(\bar{v}) = \bar{b}$ , then  $\bar{a}$  and  $\bar{b}$  are horizontal vectors at  $T_{F(\bar{p})} \widetilde{PSL}_2(\mathbb{R}, \tau)$ , hence  $d\pi(\bar{a}) = a \in T_q \mathbb{M}^2$  and  $d\pi(\bar{b}) = b \in T_q \mathbb{M}^2$ , where  $q = \pi(F(\bar{p}))$ .

Now, consider  $f$  an isometry of  $M^2$ , such that,  $f(p) = q$ ,  $df(u) = a$ , and  $df(v) = b$ . Then,

$$\begin{aligned} \langle d\pi(dF(\bar{u})), d\pi(dF(\bar{v})) \rangle_{\widetilde{PSL}_2(\mathbb{R}, \tau)} &= \langle d\pi(\bar{a}), d\pi(\bar{b}) \rangle_{\widetilde{PSL}_2(\mathbb{R}, \tau)} \\ &= \langle a, b \rangle_{M^2} \\ &= \langle u, v \rangle_{M^2}. \\ &= \langle df(a), df(b) \rangle_{M^2} \end{aligned}$$

the least equality holds since that  $f : M^2 \rightarrow M^2$ ,  $f(p) = q$  is an isometry, so we have  $F(z, t) = (f(z), h(z, t))$ .

**Proposition 5.1.1.** The isometries of  $\widetilde{PSL}_2(\mathbb{R}, \tau)$  are given by,  
In the half-plane model for the hyperbolic space  $M^2$

$$F(z, t) = (f(z), t - 2\tau \arg f' + c)$$

or

$$G(z, t) = (-\overline{f}(z), -t + 2\tau \arg f' + c)$$

where  $f$  is a positive isometry of  $\mathbb{H}^2$  and  $c$  is a real number.

In the disk model,

$$F(z, t) = (f(z), t - 2\tau \arg f' + c)$$

or

$$G(z, t) = (\overline{f}(z), -t + 2\tau \arg f' + c)$$

where  $f$  is a positive isometry of  $\mathbb{D}^2$  and  $c$  is a real number.

*Proof.* We will consider the half-plane model. The proof for the disk model is analogous. As  $F$  is a isometry of  $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$ , we must have,

$$F^* (\lambda^2(z)|dz|^2 + (-\tau\lambda(z)(dz + d\overline{z}) + dt)^2) = \lambda^2(z)|dz|^2 + (-\tau\lambda(z)(dz + d\overline{z}) + dt)^2$$

since,

$$f^*(\lambda^2|dz|^2) = \lambda^2(z)|dz|^2$$

we have

$$F^*(-\tau\lambda(z)(dz + d\overline{z}) + dt) = \pm(-\tau\lambda(z)(dz + d\overline{z}) + dt).$$

We suppose first that

$$F^*(-\tau\lambda(z)(dz + d\overline{z}) + dt) = +(-\tau\lambda(z)(dz + d\overline{z}) + dt). \quad (5-1)$$

If  $f$  is a positive isometry of  $\mathbb{H}^2$ , then

$$(5-1) \Leftrightarrow -\tau\lambda(f(z))(df(z) + d\overline{f(z)}) + dh = -\tau\lambda(z)(dz + d\overline{z}) + dt$$

$$\Leftrightarrow -\tau\lambda(f(z))(f'dz + \overline{f'}d\overline{z}) + dh = -\tau\lambda(z)(dz + d\overline{z}) + dt$$

$$\Leftrightarrow -\tau\lambda(f(z))(f'dz + \overline{f'}d\overline{z}) + h_z dz + h_{\overline{z}} d\overline{z} + h_t dt = -\tau\lambda(z)(dz + d\overline{z}) + dt$$

$$\Leftrightarrow \begin{cases} -\tau\lambda(f(z))f' + h_z = -\tau\lambda(z); \\ -\tau\lambda(f(z))\overline{f'} + h_{\overline{z}} = -\tau\lambda(z); \\ h_t = 1; \end{cases}$$

$$\Leftrightarrow \begin{cases} -\tau\lambda(f(z))f' + h_z = -\tau\lambda(z); \\ h_t = 1; \end{cases}$$

Consequently,  $h$  is a function of the form  $h(z, t) = \varphi(z) + t$ , where  $\varphi$  is a real function that verifies,

$$\begin{aligned}\varphi_z(z) &= \tau(\lambda(f(z))f' - \lambda(z)) \\ &= \tau \left[ \frac{2if'}{f - \bar{f}} - \frac{2i}{z - \bar{z}} \right] \\ &= 2\tau i [\log(f - \bar{f}) - \log(z - \bar{z})]_z \\ \Leftrightarrow \varphi &= 2\tau i \log \left( \frac{f - \bar{f}}{z - \bar{z}} \right) + \bar{\psi}\end{aligned}$$

where  $\bar{\psi}$  is a holomorphic function. By other hand, if  $f$  is a positive isometry of  $\mathbb{H}^2$ , then

$$f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

A simple computation gives,

$$\frac{f - \bar{f}}{z - \bar{z}} = \frac{1}{|cz + d|^2} = |f'(z)|$$

so, we obtain,

$$\varphi = 2\tau i \arg |f'(z)| + \bar{\psi}$$

as  $\psi$  is holomorphic and  $\varphi$  is a real function, we must have

$$\psi = 2\tau i \log(f') + c$$

where  $c$  is a constant real. So we conclude that

$$\varphi = -2\tau \arg(f'(z)) + c.$$

Thus

$$h(z, t) = t - 2\tau \arg(f'(z)) + c.$$

Now let  $f$  be a negative isometry, that is  $f = -\bar{g}$  where  $g$  is a positive isometry of  $\mathbb{H}^2$ . Thus, we have,

$$\begin{aligned}(5 - 1) \Leftrightarrow & -\tau \lambda(f(z))(df(z) + \overline{df(z)}) + dh = -\tau \lambda(z)(dz + d\bar{z}) + dt \\ \Leftrightarrow & \tau \lambda(f(z))(d\bar{g}(z) + \overline{dg(z)}) + dh = -\tau \lambda(z)(dz + d\bar{z}) + dt \\ \Leftrightarrow & \tau \lambda(f(z))(\bar{g}' d\bar{z} + g' d\bar{z}) + h_z dz + h_{\bar{z}} d\bar{z} + h_t dt = -\tau \lambda(z)(dz + d\bar{z}) + dt\end{aligned}$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = -\tau\lambda(z); \\ \tau\lambda(f(z))\bar{g}' + h_{\bar{z}} = -\tau\lambda(z); \\ h_t = 1; \end{cases}$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = -\tau\lambda(z); \\ h_t = 1; \end{cases}.$$

Again  $h$  is of the form  $h(z, t) = \varphi(z) + t$ , where  $\varphi$  is a real function that verifies,

$$\begin{aligned} \varphi_z &= -\tau(\lambda(f(z))g' + \lambda(z)) \\ &= -\tau \left[ \frac{2ig'}{g - \bar{g}} + \frac{2i}{z - \bar{z}} \right] \\ &= -2\tau i [\log(g - \bar{g}) + \log(z - \bar{z})]_z \end{aligned}$$

so,

$$\varphi = -2\tau i [\log(g - \bar{g}) + \log(z - \bar{z})] + \bar{\psi}$$

where  $\psi$  is holomorphic. Since  $\varphi$  is real, this implies that  $[\log(g - \bar{g}) + \log(z - \bar{z})]$  is harmonic, which is false. Thus  $f$  must be a positive function.

Now, we suppose that  $F$  verify,

$$F^*(-\tau\lambda(z)(dz + d\bar{z}) + dt) = -(-\tau\lambda(z)(dz + d\bar{z}) + dt) \quad (5-2)$$

Again, we consider a negative isometry, that is, we consider  $f = -\bar{g}$  where  $g$  is a positive isometry of  $\mathbb{H}^2$ , so

$$(5-2) \Leftrightarrow -\tau\lambda(f(z))(df(z) + d\overline{f(z)}) + dh = -(-\tau\lambda(z)(dz + d\bar{z}) + dt)$$

$$\Leftrightarrow -\tau\lambda(f(z))(-d\bar{g}(z) - d\overline{g(z)}) + dh = -(-\tau\lambda(z)(dz + d\bar{z}) + dt)$$

$$\Leftrightarrow \tau\lambda(f(z))(\bar{g}'d\bar{z} + g'd\bar{z}) + dh = -(-\tau\lambda(z)(dz + d\bar{z}) + dt)$$

$$\Leftrightarrow \tau\lambda(f(z))(\bar{g}'d\bar{z} + g'd\bar{z}) + h_z dz + h_{\bar{z}} d\bar{z} + h_t dt = \tau\lambda(z)(dz + d\bar{z}) - dt$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = \tau\lambda(z); \\ \tau\lambda(f(z))\bar{g}' + h_{\bar{z}} = \tau\lambda(z); \\ h_t = -1; \end{cases}$$

$$\Leftrightarrow \begin{cases} \tau \lambda(f(z))g' + h_z = \tau \lambda(z); \\ h_t = -1; \end{cases}$$

Again  $h$  is of the form  $h(z, t) = \varphi(z) - t$ , where  $\varphi$  is a real function verify

$$\begin{aligned} \varphi_z &= -\tau [\lambda(f(z))g' - \lambda(z)] \\ &= -\tau \left[ \frac{2ig'}{g - \bar{g}} - \frac{2i}{z - \bar{z}} \right] \\ &= -2\tau i [\log(g - \bar{g}) - \log(z - \bar{z})]_z \end{aligned}$$

so,

$$\varphi = -2\tau i \log \left( \frac{g - \bar{g}}{z - \bar{z}} \right) + \bar{\psi}$$

where  $\bar{\psi}$  is a holomorphic function. On the other hand, making the same calculation as above, we obtain

$$\varphi = -2\tau i \log |g'(z)| + \bar{\psi}$$

as  $\psi$  is holomorphic and  $\varphi$  is a real function, we must have

$$\psi = -2\tau i \log(g') + c$$

where  $c$  is a constant real. So we conclude that

$$\varphi = 2\tau \arg(g'(z)) + c$$

Thus

$$h(z, t) = -t + 2\tau \arg(f'(z)) + c$$

Finally, in this case, if we consider an isometry positive  $f$  we get a contradiction.  $\square$

## 5.2

### The mean curvature equation in $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$

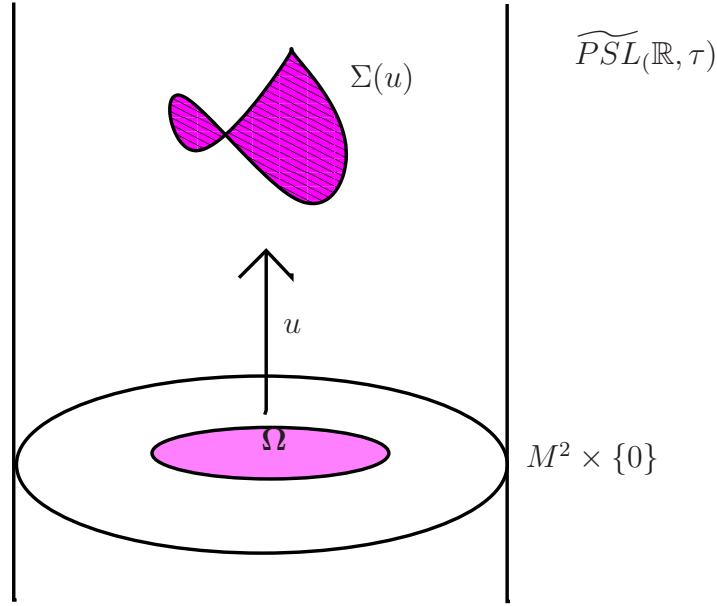
In this section we will explore the equation of the mean curvature in the divergence form.

Recall that, for a Riemannian submersion we have the notion of graph.

**Definition 5.2.1.** A graph in  $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$  over a domain  $\Omega$  of  $M^2$  is the image of a section  $s_0 : \Omega \subset M^2 \longrightarrow \widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$ .

Given a domain  $\Omega \subset M^2$  we also denote by  $\Omega$  its lift to  $M^2 \times \{0\}$ , with this identification we have that the graph  $\Sigma(u)$  of  $u \in (C^0(\partial\Omega) \cap C^\infty(\Omega))$  is given by (see figure),

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{\mathbf{PSL}}_2(\mathbb{R}, \tau); (x, y) \in \Omega\}$$



**Lemma 5.2.1.** *Let  $\Sigma(u)$  be a graph in  $\widetilde{\mathbf{PSL}}_2(\mathbb{R}, \tau)$  of the function*

$$u : \Omega \subset M^2 \longrightarrow \mathbb{R}$$

*having constant mean curvature  $H$ . Then, the function  $u$  satisfies the equation*

$$2H = \operatorname{div}_{M^2} \left( \frac{\alpha}{W} e_1 + \frac{\beta}{W} e_2 \right),$$

*where  $W = \sqrt{1 + \alpha^2 + \beta^2}$  and,*

$$- \alpha = \frac{u_x}{\lambda} + 2\tau \frac{\lambda_y}{\lambda^2},$$

$$- \beta = \frac{u_y}{\lambda} - 2\tau \frac{\lambda_x}{\lambda^2}.$$

*Proof.* The proof follows directly from Lema 4.1.2. □

Taking into account the notations of the Lemma 5.2.1 we obtain the next Proposition.

**Proposition 5.2.1.** *By expanding the equation of the mean curvature from the equation of divergence form, we obtain*

$$2H\lambda^2W^3 = \lambda\alpha_x(1+\beta^2) + \lambda\beta_y(1+\alpha^2) - \lambda\alpha\beta(\alpha_y + \beta_x) + (\lambda_x\alpha + \lambda_y\beta)W^2 \quad (5-3)$$

where,  $W^2 = 1 + \alpha^2 + \beta^2$ , and

$$\begin{aligned} -\alpha_x &= \frac{1}{\lambda^2} \left[ u_{xx}\lambda - u_x\lambda_x + \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{xy} - 2\lambda\lambda_x\lambda_y) \right] \\ -\alpha_y &= \frac{1}{\lambda^2} \left[ u_{xy}\lambda - u_x\lambda_y + \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{yy} - 2\lambda\lambda_y^2) \right] \\ -\beta_x &= \frac{1}{\lambda^2} \left[ u_{xy}\lambda - u_y\lambda_x - \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{xx} - 2\lambda\lambda_x^2) \right] \\ -\beta_y &= \frac{1}{\lambda^2} \left[ u_{yy}\lambda - u_y\lambda_y - \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{xy} - 2\lambda\lambda_x\lambda_y) \right] \end{aligned}$$

*Proof.* Since  $M^2$  has conformal metric to  $\mathbb{R}^2$  we can use the formula,

$$\text{div}_{M^2} \left( \frac{\alpha}{\lambda W} \partial_x + \frac{\beta}{\lambda W} \partial_y \right) = \frac{1}{\lambda^2} \text{div}_{\mathbb{R}^2} \left( \lambda^2 \left( \frac{\alpha}{\lambda W} \partial_x + \frac{\beta}{\lambda W} \partial_y \right) \right)$$

we obtain,

$$\begin{aligned} 2H\lambda^2 &= \text{div}_{\mathbb{R}^2} \left( \frac{\lambda}{W} (\alpha \partial_x + \beta \partial_y) \right) \\ &= \left( \left( \frac{\lambda}{W} \right)_x \partial_x + \left( \frac{\lambda}{W} \right)_y \partial_y \right) (\alpha \partial_x + \beta \partial_y) + \\ &\quad + \frac{\lambda}{W} \text{div}_{\mathbb{R}^2} (\alpha \partial_x + \beta \partial_y) \\ &= \alpha \left( \frac{\lambda}{W} \right)_x + \beta \left( \frac{\lambda}{W} \right)_y + \frac{\lambda}{W} (\alpha_x + \beta_y) \end{aligned}$$

Observe that:

$$\begin{aligned} \left( \frac{\lambda}{W} \right)_x &= \frac{\lambda_x W - \lambda W_x}{W^2} \\ \left( \frac{\lambda}{W} \right)_y &= \frac{\lambda_y W - \lambda W_y}{W^2} \end{aligned}$$

and

$$\begin{aligned} w_x &= \frac{\alpha\alpha_x + \beta\beta_x}{W} \\ w_y &= \frac{\alpha\alpha_y + \beta\beta_y}{W} \end{aligned}$$

by substitution, we obtain:

$$\begin{aligned} 2H\lambda^2 &= \frac{\alpha}{W^2} \left( \lambda_x W - \frac{\lambda}{W} (\alpha\alpha_x + \beta\beta_x) \right) + \frac{\beta}{W^2} \left( \lambda_y W - \frac{\lambda}{W} (\alpha\alpha_y + \beta\beta_y) \right) \\ &\quad + \frac{\lambda}{W^3} (\lambda_x W^2 + \beta_y W^2) \end{aligned}$$



which we be write in the form:

$$\begin{aligned} 2H\lambda^2W^3 &= \alpha\lambda_xW^2 - \lambda\alpha(\alpha\alpha_x + \beta\beta_x) + \beta\lambda_yW^2 + \\ &\quad + \lambda(\alpha_xW^2 + \beta_yW^2) - \lambda\beta(\alpha\alpha_y + \beta\beta_y) \end{aligned}$$

remember that  $W^2 = 1 + \alpha^2 + \beta^2$ , so

$$\begin{aligned} 2H\lambda^2W^3 &= \alpha\lambda_x(1 + \alpha^2 + \beta^2) - \lambda\alpha^2\alpha_x - \lambda\alpha\beta\beta_x + \beta\lambda_y(1 + \alpha^2 + \beta^2) + \\ &\quad - \lambda\alpha\beta\alpha_y - \lambda\beta^2\beta_y + \lambda\alpha_x(1 + \alpha^2 + \beta^2) + \lambda\beta_y(1 + \alpha^2 + \beta^2) \\ &= \lambda\alpha_x(1 + \beta^2) + \lambda\beta_y(1 + \alpha^2) - \lambda\alpha\beta(\alpha_y + \beta_x) + \\ &\quad + \lambda_x\alpha(1 + \alpha^2 + \beta^2) + \lambda_y\beta(1 + \alpha^2 + \beta^2) \end{aligned}$$

This gives,

$$2H\lambda^2W^3 = \lambda\alpha_x(1 + \beta^2) + \lambda\beta_y(1 + \alpha^2) - \lambda\alpha\beta(\alpha_y + \beta_x) + (\lambda_x\alpha + \lambda_y\beta)W^2$$

A simple computation gives the other expressions.

□

An immediate consequence from Proposition 5.2.1 is the next corollary. Here we are considering the half-plane model for the hyperbolic space, that is  $M^2 \equiv \mathbb{H}^2$ .

**Corollary 5.2.1.** *By expanding the mean curvature equation of the divergence form in  $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$ , we obtain:*

$$\begin{aligned} 2H\lambda^2m^3 &= u_{xx}(\lambda^3 + \lambda u_y^2) + u_{yy}\lambda(\lambda^2 + (u_x - 2\tau\lambda)^2) - 2u_{xy}\lambda(u_x - 2\tau\lambda)u_y + \\ &\quad - u_xu_y\lambda^2(u_x - 2\tau\lambda) - \lambda^2u_y^3 \end{aligned}$$

$$\text{where } m = \sqrt{\lambda^2 + (2\tau\lambda - u_x)^2 + u_y^2}$$

*Proof.* Since  $\lambda = \frac{1}{y}$ , so  $\lambda_x \equiv 0$ . The equation from the Proposition 5.2.1 becomes:

$$\begin{aligned} 2H\lambda^2W^3 &= \lambda\alpha_x(1 + \beta^2) + \lambda\beta_y(1 + \alpha^2) - \lambda\alpha\beta(\beta_x + \alpha_y) + \\ &\quad - \lambda^2(\beta + \alpha^2\beta + \beta^3) \end{aligned}$$

By considering:

$$\alpha = \frac{u_x}{\lambda} - 2\tau, \quad \beta = \frac{u_y}{\lambda}, \quad \lambda = \frac{1}{y}$$

$$\begin{aligned}\alpha_x &= \frac{u_{xx}}{\lambda}, & \lambda_y &= u_x + \frac{u_{xy}}{\lambda} \\ \beta_x &= \frac{u_{xy}}{\lambda}, & \lambda_y &= u_y + \frac{u_{xy}}{\lambda}\end{aligned}$$

we obtain:

$$\begin{aligned}2H\lambda^2W^3 &= u_{xx} \left(1 + \frac{u_y^2}{\lambda^2}\right) + (\lambda u_y + u_{yy}) \left(1 + \frac{(u_x - 2\tau\lambda)^2}{\lambda^2}\right) + \\ &\quad -\lambda \left(\frac{u_x}{\lambda} - 2\tau\right) \left(\frac{u_y}{\lambda}\right) \left(u_x + 2\frac{u_{xy}}{\lambda}\right) - \lambda^2 \left(\frac{u_y}{\lambda} + \frac{u_y}{\lambda} \left(\frac{u_x}{\lambda} - 2\tau\right)^2 + \frac{u_y^3}{\lambda^3}\right) \\ &= u_{xx} \left(\frac{\lambda^2 + u_y^2}{\lambda^2}\right) + u_{yy} \left(\frac{\lambda^2 + (u_x - 2\tau\lambda)^2}{\lambda^2}\right) + \\ &\quad + \lambda u_y \left(\frac{\lambda^2 + (u_x - 2\tau\lambda)^2}{\lambda^2}\right) - 2u_{xy} \left(\frac{u_x - 2\tau\lambda}{\lambda}\right) \frac{u_y}{\lambda} \\ &\quad - u_x u_y \left(\frac{u_x - 2\tau\lambda}{\lambda}\right) - \lambda^2 \left(\frac{u_y}{\lambda} + \frac{u_y}{\lambda} \left(\frac{(u_x - 2\tau\lambda)^2}{\lambda^2}\right) + \frac{u_y^3}{\lambda^3}\right).\end{aligned}$$

Since  $\lambda^3W^3 = m^3$ , where  $m = \sqrt{\lambda^2 + (u_x - 2\tau\lambda)^2 + u_y^2}$  we obtain:

$$\begin{aligned}2H\lambda^2m^3 &= u_{xx}(\lambda^3 + \lambda u_y^2) + u_{yy}\lambda(\lambda^2 + (u_x - 2\tau\lambda)^2) + \\ &\quad - 2u_{xy}\lambda(u_x - 2\tau\lambda)u_y - u_x u_y \lambda^2(u_x - 2\tau\lambda) + \\ &\quad - \lambda^5 \left(\frac{u_y}{\lambda} + \frac{u_y}{\lambda} \frac{(u_x - 2\tau\lambda)^2}{\lambda^2} + \frac{u_y^3}{\lambda^3}\right) + \lambda^2 u_y (\lambda^2 + (u_x - 2\tau\lambda)^2) \\ &= u_{xx}(\lambda^3 + \lambda u_y^2) + u_{yy}\lambda(\lambda^2 + (u_x - 2\tau\lambda)^2) - 2u_{xy}\lambda(u_x - 2\tau\lambda)u_y + \\ &\quad - u_x u_y \lambda^2(u_x - 2\tau\lambda) - \lambda^2 u_y^3\end{aligned}$$

This complete the proof. □

On the other hand, by considering the coefficients of the first and second fundamental form of a surface immersed into  $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$  we can obtain the mean curvature equation. For example, taking the half plane model  $M^2$  for the hyperbolic space. Taking graphs of the form  $t = u(x, y)$ , where  $u$  is a smooth function. Setting  $S = \text{graf}(u) \in \widetilde{\text{PSL}}_2(\mathbb{R})$ , and parameterizing  $S$  by

$$\varphi(x, y) = (x, y, u(x, y))$$

The coordinate global frame field to the graph is given by,

$$\begin{cases} \varphi_x = \partial_x + u_x \partial_t = \lambda E_1 + (u_x - 2\tau\lambda) E_3; \\ \varphi_y = \partial_y + u_y \partial_t = \lambda E_2 + u_y E_3; \end{cases}$$

So the normal vector is given by,

$$N = \frac{1}{\sqrt{(u_x - 2\tau\lambda)^2 + \lambda^2 + u_y^2}} [-(u_x - 2\tau\lambda) E_1 - u_y E_2 + \lambda E_3]$$

Set,

$$m = m(x, t) = \sqrt{(u_x - 2\tau\lambda)^2 + \lambda^2 + u_y^2}.$$

**Lemma 5.2.2.** *With the notations above, and denoting by  $H$  the mean curvatura of  $S$ , then  $H$  satisfies*

$$\begin{aligned} 2H\lambda^2 m^3 &= u_{xx}(\lambda^3 + \lambda u_y^2) + u_{yy}\lambda(\lambda^2 + (u_x - 2\tau\lambda)^2) - 2u_{xy}\lambda(u_x - 2\tau\lambda)u_y + \\ &\quad - u_x u_y \lambda^2 (u_x - 2\tau\lambda) - \lambda^2 u_y^3 \end{aligned}$$

*Proof.* The coefficients of the second fundamental forms are given by,

$$\begin{aligned} b_{11} &= \langle \overline{\nabla}_{\varphi_x} \varphi_x, N \rangle, & g_{11} &= \langle \varphi_x, \varphi_x \rangle \\ b_{12} &= \langle \overline{\nabla}_{\varphi_x} \varphi_y, N \rangle, & g_{12} &= \langle \varphi_x, \varphi_y \rangle \\ b_{22} &= \langle \overline{\nabla}_{\varphi_y} \varphi_y, N \rangle, & g_{22} &= \langle \varphi_y, \varphi_y \rangle \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is the metric of  $\widetilde{\text{PSL}}_2(\mathbb{R})$ , then  $H$  satisfies,

$$2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2} \quad (5-4)$$

It is easily deduce that the connection is given by,

$$\begin{aligned} \overline{\nabla}_{\varphi_x} \varphi_x &= (\lambda^2 - 2\tau\lambda(u_x - 2\tau\lambda))E_2 + u_{xx}E_3 \\ \overline{\nabla}_{\varphi_x} \varphi_y &= (\lambda\tau(u_x - 2\tau\lambda) - \lambda^2)E_1 - \lambda\tau u_y E_2 + (u_{xy} + \lambda^2\tau)E_3 \\ \overline{\nabla}_{\varphi_y} \varphi_y &= 2\tau\lambda u_y E_1 - \lambda^2 E_2 + u_{yy}E_3 \end{aligned}$$

and with this,

$$\begin{aligned} b_{11} &= \lambda u_{xx} - u_y(\lambda^2 - 2\tau\lambda(u_x - 2\tau\lambda)) \\ b_{12} &= \lambda(u_{xy} + \lambda^2\tau) - (u_x - 2\tau\lambda)(\lambda\tau(u_x - 2\tau\lambda) - \lambda^2) + \lambda\tau u_y^2 \\ b_{22} &= \lambda u_{yy} - 2\tau\lambda u_y(u_x - 2\tau\lambda) + \lambda^2 u_y \end{aligned}$$

since,

$$\begin{aligned} g_{11} &= \lambda^2 + (u_x - 2\tau\lambda)^2 \\ g_{12} &= u_y(u_x - 2\tau\lambda) \\ g_{22} &= \lambda^2 + u_y^2 \end{aligned}$$

by substitution this expressions in (5-4), we obtain the lemma.  $\square$

### 5.3

#### Maximum principle in $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$

An important criterium in Riemannian Geometry is the maximum principle. There are many books, which study the maximum principle. We enunciate this principle in the next form,

**Theorem 5.3.1.** (17, Theorem 3.1)[Maximum principle] *Let  $\Sigma_1$  and  $\Sigma_2$  two convex surfaces in  $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$ , such that  $\Sigma_2$  touch  $\Sigma_1$  at  $p \in \Sigma_1$ , that is  $p \in \Sigma_1 \cap \Sigma_2$ , and suppose that, there is a neighborhood of  $p$ , such that,  $\Sigma_2$  stay in the mean convex side of  $\Sigma_1$ . Denoting by  $\vec{H}_{\Sigma_1}$ , and  $\vec{H}_{\Sigma_2}$ , the mean curvature vector field respectively, if:*

$$|\vec{H}_{\Sigma_1}| = |\vec{H}_{\Sigma_2}| = cte$$

and

$$\langle \vec{H}_{\Sigma_1}(p), \vec{H}_{\Sigma_2}(p) \rangle \geq 0.$$

Then,  $\Sigma_1 = \Sigma_2$ . When the intersection point  $p$  belongs to the boundary of the surfaces, the result holds as well, provided further that the two boundaries are tangent and both are local graphs over a common neighborhood in  $T_p\Sigma_1 = T_p\Sigma_2$ .

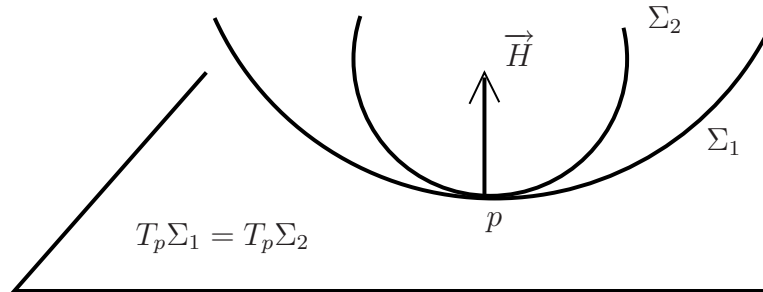


Figure 5.1: Schematic figure for the maximum principle.

The proof of the maximum principle is based on the fact that a constant mean curvature surface in  $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$  locally satisfies a second order elliptic

PDE, see Lemma 5.2.2. For the proof of the maximum principle in space forms see (17, Theorem 3.1). The proof generalizes to  $\widetilde{PSL}_2(\mathbb{R}, \tau)$  as well.

**Remark 5.3.1.** *As simple application, we will show that, there is no entire  $H$ -graph  $G$  in  $\widetilde{PSL}_2(\mathbb{R}, \tau)$  having constant mean curvature  $H > \frac{1}{2}$ . From example 6.2.1, we know that, there are rotational spheres having constant mean curvature  $H > \frac{1}{2}$ . We denote this sphere by  $S$ .*

*Since the vertical translations are isometries on  $\widetilde{PSL}_2(\mathbb{R}, \tau)$ . Moving the sphere  $S$ , such that  $S$  lies in the mean convex side of  $G$  and touch  $G$  at  $p \in S$ , then by using the maximum principle, we obtain  $G \equiv S$ , which is a contradiction.*