# 4 Correspondence between normal Natural Deduction and cut-free Sequent Calculus

## 4.1 Introduction

Logic has a strong syntactical and deductive tradition, semantics is relatively new in logic. From the model-theoretic point of view there might be many approaches to provide semantics. Algebras, categories and Tarski-based semantics are some examples. There is also proof-theoretical semantics. The Curry-Howard isomorphism can be seen as one of the most well-known representatives of this kind of semantics. Categorical models can be also considered as representants of this proof-theoretical approach. However, even for the most well-known propositional logics, proof-theoretical semantics faces some problems. Natural Deduction and Sequent Calculus are mostly taken into account when discussing such problems. One of the points that deserve special attention is the (potential?) isomorphism between both systems. When considering normal and cut-free proofs, the literature has reported some problems (see next section).

This chapter and the next are dedicated to the definition of isomorphic translations between Natural Deduction and Sequent Calculus. In this chapter, we deal only with translations between normal and cut-free derivations, showing that these translations are in a one-to-one correspondence.

In next chapter we define translations between any derivations and not only normal and cut-free ones. With isomorphism, "cut-free proofs in Sequent Calculus and normal proofs in Natural Deduction became mere notational variants of one and the same proof" (15).

## 4.2 Motivation

In the first part, we extended Segeberg's general completeness proof for propositional logic to finitely-valued propositional logics (19).

For this purpose, we followed Segeberg's original idea of defining a natural deductive system whose rules correspond to rows of truth-tables, but instead of having $n$ types of rules (one for each truth-value), we use a bivalent representation of many-valued truth-tables that makes use of a technique defined by Caleiro and João Marcos (6).

Although the idea is quite simple, it enables us to find a Natural Deduction system for any finite-valued propositional logic. However, the system defined has so many rules it might be laborious to work with it. We believe that a Sequent Calculus system defined in a similar way would be more intuitive.

This idea made us think about translations between Sequent Calculus and Natural Deduction. Since the firsts definitions of Natural Deduction and Sequent Calculus systems in 1934, translation between the system were defined preserving conclusion and hypothesis. These translations do not preserve normal and cut-free derivations. For example:

**Gentzen** In the same paper (8) in which Gentzen defined the firsts systems of Natural Deduction ($\mathcal{NK}$ and $\mathcal{NJ}$) and Sequent Calculus ($\mathcal{LK}$ and $\mathcal{LJ}$) for predicate logic, Gentzen also showed that these systems are equivalent. The translations defined in order to show the equivalence map normal Natural Deduction derivations into Sequent Calculus derivations with cut.

The systems as defined by Gentzen are not isomorphic. There is, for instance, only one normal $\mathcal{NJ}$-derivation of $(A \wedge B) \rightarrow (A \vee B)$ but two cut-free $\mathcal{LJ}$-derivations. Besides that, as interchange and contraction can be applied at any step of a derivation, there may be a large number of $\mathcal{LJ}$-derivations that is the image of the same $\mathcal{NJ}$-derivation. So, one of the systems or both would have to me modified, in some extent, to achieve isomorphism.

**Zucker** In (21), Zucker relates cut-elimination with reduction, but only for the fragment $\{\wedge, \rightarrow, \forall, \perp\}$ of intuitionistic predicate logic. He presents an example of a non-terminating and non-repeating reduction sequence of the Sequent Calculus if disjunction is added to the system. The translation defined between normal and cut-free derivations is not an isomorphism.

In Zucker's Sequent Calculus's system $S$, the formulas in the antecedent of a sequent form a set, and not a sequence as in $\mathcal{LJ}$ and $\mathcal{LK}$. This means we do not need to worry about exchange and thinning rules. Also, the premisses are indexed. In Natural Deduction, more than one formula of the same form can be discharged in the application of a single rule. The idea of the indexed formulas in $S$ is that, in a transformation, all the formulas discharged in a single rule in Natural Deduction would be mapped to the same index in $S$ (and conversely).

The premisses of Zucker's Natural Deduction are also indexed, but those indices are not part of the formal system, they are used only in the metalevel to facilitate the definition of the mapping between the systems.

Reducing a derivation in Natural Deduction means that the resulting derivation is either normal or the complexity of at least one of its maximum formulas is reduced. However, conversions in Sequent Calculus may only mean the permutation of a cut application one step up. With this in mind, it is easy to see that reduction sequences in Sequent Calculus are usually larger than reduction sequences in Natural Deduction. To deal with this situation, Zucker defined an equivalence that comprises these permutative conversions.

Pottinger (17), improved Zucker's method by simplifying it and extending it to the full intuitionistic propositional logic.

**Danos, Joinet and Schellinx** Danos, Joinet and Schellinx (7) have an isomorphism between Sequent Calculus and Natural Deduction passing through Linear Logic.

**Negri and von Plato** Negri and von Plato's translation defined in (15) is between Sequent Calculus with independent contexts and Natural Deduction in Sequent Calculus style with general elimination rules. They show that weakening can be related to *vacuous* discharge of assumptions and that contraction can be related to *multiple* discharge.

The translation takes into account only Sequent Calculus derivations in which the principal formulas in weakening and contraction are *used*, i.e., they are the principal formulas of a left rule. This means that, for instace, if there exists a derivation of $A$ from $\Gamma$ in Sequent Calculus, a derivation of $A$ from $\Gamma, B$, where $B \notin \Gamma$, has no equivalence in Natural Deduction.

To give an example of a derivation whose translation as defined in (15) does not work, let us take the implication $A \to (B \to (B \to (A \to B)))$. The derivation of this implication in Sequent Calculus has three applications of the weakening rule, which can be applied in different levels of the derivations. For example, with the notation used in (15):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,Wk}{A, A, B \Rightarrow B}\,Wk}{A, A, B, B \Rightarrow B}\,Wk}{A, B, B \Rightarrow A \supset B}\,R\supset}{A, B \Rightarrow B \supset (A \supset B)}\,R\supset}{A \Rightarrow B \supset (B \supset (A \supset B))}\,R\supset}{\Rightarrow A \supset (B \supset (B \supset (A \supset B)))}\,R\supset
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,Wk}{B \Rightarrow A \supset B}\,R\supset}{\Rightarrow B \supset (A \supset B)}\,R\supset}{B \Rightarrow B \supset (A \supset B)}\,Wk}{\Rightarrow B \supset (B \supset (A \supset B))}\,R\supset}{A \Rightarrow B \supset (B \supset (A \supset B))}\,Wk}{\Rightarrow A \supset (B \supset (B \supset (A \supset B)))}\,R\supset
$$

Accordin to the translation defined in (15), the derivation on the left side has no correspondent in Natural Deduction. and the derivation on the right side corresponds to the following derivation in Natural Deduction:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{2.}{[B]}}{A \supset B}\,I\supset,1.}{B \rightarrow (A \supset B)}\,I\supset,2.}{B \supset (B \rightarrow (A \supset B))}\,I\supset,3.}{A \supset (B \supset (B \rightarrow (A \supset B)))}\,I\supset,4.
$$

where 1., 3., and 4., are "ghost" labels that correspond to vacuous discharge. In the systems we are going to work with there is only one possible (normal/cut-free) derivation of $A \rightarrow (B \rightarrow (B \rightarrow (A \rightarrow B)))$ in Sequent Calculus and in Natural Deduction (see section 4.5). The translations defined in this chapter work for every cut-free derivation in Sequent Calculus and every normal derivation in Natural Deduction.

**Focuses proofs** Nigam and Miller (16) showed that different proof systems, including Natural Deduction and Sequent Calculus, have the same provable sets of formulas by showing that each system can be encoded into a Focused Linear Logic system. In (10), Henriksen showed that Linear Logic is not needed and showed a similar result from that of (16) by encoding the systems into a focused intuitionistic system. Negri and von Plato (15) showed the relation between structural rules in Sequent Calculus and discharge of formulas in Natural Deduction.

## 4.3 Sequent Calculus and Natural Deduction

There are propositions with more possible derivations in Sequent Calculus than in Natural Deduction. For instance, we have two possible derivations for the proposition $(A \wedge B) \rightarrow (A \vee C)$ in the Sequent Calculus system defined in (9):

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A \vee C}\mathcal{R}1\vee}{A \wedge B \vdash A \vee C}\mathcal{L}1\wedge}{\vdash (A \wedge B) \rightarrow (A \vee C)}\mathcal{R}\rightarrow \qquad \dfrac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\mathcal{L}1\wedge}{A \wedge B \vdash A \vee C}\mathcal{R}1\vee}{\vdash (A \wedge B) \rightarrow (A \vee C)}\mathcal{R}\rightarrow$$

and only one in the Natural Deduction system defined in (18):

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{A\&B}}{A}}{A \vee C}}{(A\&B) \rightarrow (A \vee C)}\,(1)$$

Thus, to define an isomorphism, we need to choose a more restricted Sequent Calculus and/or a more liberal Natural Deduction system.

### 4.3.1
### Sequent Calculus



For the Sequent Calculus, we decided to work with the stoup-based system LJT. LJT is the implicational fragment of LKT which was first introduced in Joinet's thesis (13). In fact, the system introduced by Joinet is a slight different version of LJT, called ILU to stress that this fragment of LKT could also be seen as the intuitionistic fragment of Girard's LU.

In (11), Herbelin defined an extension of the usual $\lambda$-calculus called $\overline{\lambda}$-calculus. However, for a $\lambda$-term of the form $(\ldots(x[u_1])\ldots\ [u_k])$, the LJT image is a proof with cuts. This term is a $\overline{\lambda}$ image of the normal term $(\ldots(x\ \ u_1)\ldots u_n)$[1], but in $\overline{\lambda}$ it is not normal due to the use of explicit substitution in $\overline{\lambda}$. Thus, (11) reports a mapping between $\lambda$ and LJT that takes normal terms as those shown in $\lambda$ into derivations in LJT with cuts. In our proposed isomorphism, we avoid this by using a notion of proof equivalence and different versions of sequent calculus and natural deduction. The paper (11) only deals with the implicational fragment of intuitionistic logic, but in his thesis (12), Herbelin extends the result to the full propositional fragment of intuitionistic logic. In table 4.1 we present LJT for the full intuitionistic propositional fragment $\{\wedge, \vee, \rightarrow, \neg, \bot\}$, where negation ($\neg$) is seen as a particular case of implication in which the consequence is always a falsity.

Herbelin's version of LJT is a slightly different version of the intuitionistic fragment of LKT. The differences are: (1) the formulas that form the disjunction in $\vee\vdash$ are outside the stoup, (2) to apply the right rules the stoup

[1]$(\ldots(x\ u_1)\ldots u_n)$ is normal in $\lambda$ whenever $u_i$ is normal.

must be empty, (3) the rule $\mathcal{D}$ keeps a copy of the formula that passed to the stoup and (4) the left rules for conjunction, which are three:

$$\frac{\Gamma, \alpha; \beta \vdash \gamma}{\Gamma; \alpha \wedge \beta \vdash \gamma} \qquad \frac{\Gamma, \beta; \alpha \vdash \gamma}{\Gamma; \alpha \wedge \beta \vdash \gamma} \qquad \frac{\Gamma, \beta, \alpha; \vdash \gamma}{\Gamma; \alpha \wedge \beta \vdash \gamma}$$

Herbelin states that these rules are necessary for cut elimination, but cut elimination is only shown for the implicational fragment. The version of LJT presented here is Herbelin's version, except for the rules of left conjunction, which are like in Joinet's thesis.

$$(\to\vdash)\ \frac{\Gamma; \vdash \alpha \quad \Gamma; \beta \vdash \gamma}{\Gamma; \alpha \to \beta \vdash \gamma} \qquad\qquad (\vdash\to)\ \frac{\Gamma, \alpha; \vdash \beta}{\Gamma; \vdash \alpha \to \beta}$$

$$(\wedge\vdash)\ \frac{\Gamma; \beta \vdash \gamma}{\Gamma; \alpha \wedge \beta \vdash \gamma} \quad \frac{\Gamma; \alpha \vdash \gamma}{\Gamma; \alpha \wedge \beta \vdash \gamma} \qquad (\vdash\wedge)\ \frac{\Gamma; \vdash \alpha \quad \Gamma; \vdash \beta}{\Gamma; \vdash \alpha \wedge \beta}$$

$$(\vee\vdash)\ \frac{\Gamma, \alpha; \vdash \gamma \quad \Gamma, \beta; \vdash \gamma}{\Gamma; \alpha \vee \beta \vdash \gamma} \qquad\qquad (\vdash\vee)\ \frac{\Gamma; \vdash \alpha}{\Gamma; \vdash \alpha \vee \beta} \quad \frac{\Gamma; \vdash \beta}{\Gamma; \vdash \alpha \vee \beta}$$

$$(\bot\vdash)\ \frac{\Gamma; \alpha \vdash \gamma}{\Gamma; \bot \vdash \gamma}$$

$$Ax\ \frac{}{\Gamma; \alpha \vdash \alpha} \qquad\qquad \mathcal{D}\ \frac{\Gamma, \alpha; \alpha \vdash \gamma}{\Gamma, \alpha; \vdash \gamma}$$

Table 4.1: LJT rules

A sequent in LJT is of the form $\Gamma; \Pi \vdash \Delta$, where $\Gamma$ is a set of formulas (possibly empty) and $\Pi$ and $\Delta$ are sets of at most one formula. The place occupied by $\Pi$, that is, the place between ';' and '$\vdash$', is called *stoup* and the formula in the stoup (if any) is called *head-formula*. In a derivation, the stoup of the conclusion must be empty.

The following are examples of derivations in LJT:

$$\frac{\dfrac{\dfrac{\dfrac{}{A \wedge B; A \vdash A}\,{\scriptstyle Ax}}{A \wedge B; A \wedge B \vdash A}\,{\scriptstyle \wedge\vdash}}{A \wedge B; \vdash A}\,{\scriptstyle \mathcal{D}}}{A \wedge B; \vdash A \vee B}\,{\scriptstyle \vdash\vee} \qquad\qquad \frac{\dfrac{\dfrac{\dfrac{}{A, B; B \vdash B}\,{\scriptstyle Ax}}{A, B; \vdash B}\,{\scriptstyle \mathcal{D}}}{B; \vdash A \to B}\,{\scriptstyle \vdash\to}}{; \vdash B \to (A \to B)}\,{\scriptstyle \vdash\to}$$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (A \to B); A \vdash A}\,{\scriptstyle Ax}}{A \wedge (A \to B); A \wedge (A \to B) \vdash A}\,{\scriptstyle \wedge\vdash}}{A \wedge (A \to B); \vdash A}\,{\scriptstyle \mathcal{D}} \quad \dfrac{}{A \wedge (A \to B); B \vdash B}\,{\scriptstyle Ax}}{A \wedge (A \to B); A \to B \vdash B}\,{\scriptstyle \to\vdash}}{A \wedge (A \to B); A \wedge (A \to B) \vdash B}\,{\scriptstyle \wedge\vdash}}{A \wedge (A \to B); \vdash B}\,{\scriptstyle \mathcal{D}}$$

The following are *not* derivations in LJT:

$$\dfrac{\dfrac{A \wedge B; B \vdash A \vee B}{A \wedge B; A \wedge B \vdash A \vee B}\,\mathcal{D}}{A \wedge B; \vdash A \vee B}\,\vdash\vee$$

If we begin the proof of $A \wedge B; \vdash A \vee B$ by applying first the rule $\mathcal{D}$, we will not be able to close the derivation.

$$\dfrac{\dfrac{A, B; \vdash B}{B; \vdash A \to B}\,\vdash\to}{; \vdash B \to (A \to B)}\,\vdash\to$$

Although the top-sequent has occurrences of formulas of the same form ($B$) in both sides of the sequent, it is not a derivation, as the top-sequent is not $Ax$. To turn it into a derivation, we need to bring an occurrence of $B$ to the stoup by applying $\mathcal{D}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A; A \vdash A}\,Ax}{A; \vdash A}\,\mathcal{D} \quad \dfrac{}{A; B \vdash B}\,Ax}{A; A \to B \vdash B}\,\to\vdash}{A; A \wedge (A \to B) \vdash B}\,\wedge\vdash$$

is not a derivation, for the conclusion has a formula in the stoup.

An inference rule can be read from the conclusion to its premisses. Note that we can only apply right rules when the stoup is empty and that we can bring a formula to the stoup with the rule $\mathcal{D}$ but we cannot take a formula from the stoup. LJT has additive contexts, i.e., the same set $\Gamma$ of assumptions in the premisses of each rule, and even though we need $\mathcal{D}$ as a rule in the system.

LJT forces a focusing in the derivation. When there is a formula in the stoup, we are "forced" to apply left rules, breaking the named formula until either an atomic formula is in the stoup, in which case we have an initial sequent, or until we apply $\vee\vdash$, in which case the stoup is empty, and we can choose between applying a right rule and the rule $\mathcal{D}$, in which case the focus is back to the head-formula. When the bottommost rule applied in a cut-free derivation is a $\mathcal{D}$-rule, we can identify the sequence of applications of left rules forced by the stoup with a positive trunk in focused proofs[2].

Because of the stoup, the system admits two cuts, a *head-cut* ($C_H$) which cuts the formula in the stoup and a *middle-cut* ($C_M$) which cuts a formula outside the stoup:

$$\dfrac{\Gamma; \Delta \vdash A \quad \Gamma; A \vdash B}{\Gamma; \Delta \vdash B}\,C_H \qquad \dfrac{\Gamma; \vdash A \quad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta \vdash B}\,C_M$$

**Definition 7 (Cut-free derivation)** *We say that a derivation* $\Pi$ *is* cut-free *in LJT when there is neither applications of* $C_H$ *nor applications of* $C_M$ *in* $\Pi$.

[2]This terminology is according to (14)

The stoup reduces the quantity of cut-free derivations that we usually have in Sequent Calculus. For instance, instead of the two possible cut-free Sequent Calculus derivations of $(A \wedge B) \to (A \vee C)$ we showed, we only have one in LJT (see figure 4.1).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \wedge B; A \vdash A}\ Ax}{A \wedge B; A \wedge B \vdash A}\ \wedge\vdash}{A \wedge B; \vdash A}\ \mathcal{D}}{A \wedge B; \vdash A \vee C}\ \vdash\vee}{; \vdash (A \wedge B) \to (A \vee C)}\ \vdash\to$$

Figure 4.1: Example of a derivation in LJT

### 4.3.2
### Natural Deduction

If we decide to use Gentzen's Natural Deduction system $\mathcal{N}J$, there would be no way to distinguish derivations with more premisses than needed. For instance, derivations of $A \wedge B$ from $A \wedge B$ and of $A \wedge B$ from $A \wedge B, C$ in LJT would be translated to the same derivation in $\mathcal{N}J$.

In order to have a faithful comparison, we decided to use a representation of Natural Deduction in a Sequent Calculus style. The system ND is presented in table 4.2.

$$(E_{\to})\ \dfrac{\Gamma \vdash \alpha \to \beta \quad \Gamma \vdash \alpha}{\Gamma \vdash \beta} \qquad\qquad (I_{\to})\ \dfrac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}$$

$$(E_{\wedge})\ \dfrac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha} \qquad \dfrac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \beta} \qquad\qquad (I_{\wedge})\ \dfrac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}$$

$$(E_{\vee})\ \dfrac{\Gamma \vdash \alpha \vee \beta \quad \Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma \vdash \gamma} \qquad\qquad (I_{\vee})\ \dfrac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \qquad \dfrac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta}$$

$$(E_{\perp})\ \dfrac{\Gamma \vdash \perp}{\Gamma \vdash \gamma} \qquad\qquad Ax\ \dfrac{}{\Gamma, \alpha \vdash \alpha}$$

Table 4.2: ND rules

**Definition 8 (Major premiss)** *The premisses* $\Gamma \vdash \alpha \to \beta, \Gamma \vdash \alpha \wedge \beta,$ $\Gamma \vdash \alpha \vee \beta$ *and* $\Gamma \vdash \perp$ *of the rules* $E_{\to}$, $E_{\wedge}$, $E_{\vee}$ *and* $E_{\perp}$ *respectively are called* major premisses . *The other premisses are called* minor premisses. *The sequent* $\Gamma, \alpha \vdash \alpha$ *in Ax is called* initial sequent.

As this system is more known, we believe there is no need to go into as many explanations as we did in the case of LJT.

A cut rule in Sequent Calculus is usually mapped into a derivation with a maximal sequent in Natural Deduction, that is, a sequent which is the conclusion of an introduction rule and major premiss of an elimination rule. But LJT has two cuts: if one is translated as a derivation with a maximal sequent, what would the other cut represent in ND? Hence, we add to our system the following admissable rule, known as *substitution rule*:

$$\frac{\Gamma \vdash \alpha \quad \Gamma, \alpha \vdash \beta}{\Gamma \vdash \beta}\ \mathcal{S}$$

**Definition 9** *The sequent* $\Gamma \vdash \alpha$ *is the* major premiss of $\mathcal{S}$.

**Definition 10 (Normal derivation)** *A derivation* $\Pi$ *is* normal *in ND when there is neither a maximal sequent nor an application of substitution rule in* $\Pi$. *Besides that, no major premiss of* $\Pi$ *is the conclusion of an application of* $E_\vee$.

This restriction is due to the fact that an application of $E_\vee$ might "hide" a maximal sequent. For instance, take the derivation in figure 4.2.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma \vdash C \vee D\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ \Gamma, C \vdash A\end{array} \quad \begin{array}{c}\Pi_3\\ \Gamma, C \vdash B\end{array}}{\Gamma, C \vdash A \wedge B}\ I_\wedge \quad \begin{array}{c}\Pi_4\\ \Gamma, D \vdash A \wedge B\end{array}}{\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ E_\wedge}\ E_\vee$$

Figure 4.2: Example of a derivation in ND

As well-known, such a proof should not be considered as normal and it should be reduced first by communing the elimination rule of the disjunction with that on the conjunction and then reducing the sequence formed with the introduction and elimination rules of the conjunction.

**Definition 11 (Detour)** *A* detour *in a derivation in ND is either a maximal sequent or an application of a rule* $\mathcal{S}$.

## 4.4 Definitions

As we are restricting ourselves to normal and cut-free derivations, we are going to use the term "bijective" rather than "isomorphic". We say that ND and LJT are *bijective* when there exists transformations $t_1$ from ND to LJT and $t_2$ from LJT to ND, such that, if $\Pi$ is a cut-free derivation in LJT, then $t_1(t_2(\Pi)) = \Pi$ and if $\Pi$ is a normal derivation in ND, then $t_2(t_1(\Pi)) = \Pi$.

The translations between ND and LJT are defined by induction on the length of derivations. For that, we are going to do the cases for all possible forms a derivation can have. But, if a derivation has the form

$$\dfrac{\dfrac{\begin{matrix}\Pi' \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}\Sigma' \\ \Gamma; B \vdash C\end{matrix}}{\Gamma; A \to B \vdash C}\,{\scriptstyle \to\vdash}}{\Gamma; \vdash C}\,{\scriptstyle \mathcal{D}}$$

we have a problem, for $\begin{matrix}\Sigma' \\ \Gamma; B \vdash C\end{matrix}$ is not a derivation (remember that, in a derivation, there is no head-formula in the conclusion) and then we cannot apply the induction hypothesis (IH). Hence, to define the translations between derivations, we need to define translations between *pseudo-derivations*. Thus, our translations are based on a pair $(p, q)$ of functions where $p$ is a map between pseudo-derivations and $q$ is a map between derivations and when we define $q$, we may use $p$, and in some cases of the definition of $p$ we use $q$.

We are going to use $\Pi$ for derivations, $\Sigma$ for pseudo-derivations and $\Psi$, usually, for either. Before defining the translations, we need to define some notions.

**Definition 12** *The sequents* $\Gamma_1 \vdash A_1, \ldots, \Gamma_n \vdash A_n$ *of a derivation* $\Pi$ *in ND form a* sequence *if* $\Gamma_i \vdash A_i, 1 \leq i < n$ *is premiss of the rule of which* $\Gamma_{i+1} \vdash A_{i+1}$ *is the conclusion. In this case, if* $i < j$*, we say that* $\Gamma_i \vdash A_i$ *is* above $\Gamma_j \vdash A_j$ *and that* $\Gamma_j \vdash A_j$ *is* below $\Gamma_i \vdash A_i$*. The notion is analogous for sequences in LJT.*

**Definition 13** *The* length *of a derivation is the number of rules it contains.*

**Definition 14** *The set of* pure elimination derivations (PED) *in ND is inductively defined as follows.*

– *a derivation* $\dfrac{}{\Gamma \vdash A}\,{\scriptstyle Ax}$ *is a PED.*

– *a normal derivation* $\dfrac{\begin{matrix}\Pi \\ \Gamma \vdash A \Rightarrow B\end{matrix} \quad \begin{matrix}\Pi' \\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash B}\,{\scriptstyle E_\to}$ *is a PED if* $\Pi$ *is.*

– *a normal derivation* $\dfrac{\begin{matrix}\Pi \\ \Gamma \vdash A \wedge B\end{matrix}}{\Gamma \vdash A}\, E_\wedge$ *is a PED if* $\Pi$ *is.*

– *a normal derivation* $\dfrac{\begin{matrix}\Pi \\ \Gamma \vdash A \wedge B\end{matrix}}{\Gamma \vdash B}\, E_\wedge$ *is a PED if* $\Pi$ *is.*

– *a normal derivation* $\dfrac{\begin{matrix}\Pi \\ \Gamma \vdash A \vee B\end{matrix} \quad \begin{matrix}\Pi_1 \\ \Gamma, A \vdash C\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, B \vdash C\end{matrix}}{\Gamma \vdash C}\, E_\vee$ *is a PED if* $\Pi$ *is.*

– *a normal derivation* $\dfrac{\begin{matrix}\Pi \\ \Gamma \vdash \bot\end{matrix}}{\Gamma \vdash A}\, E_\bot$ *is a PED if* $\Pi$ *is.*

**Definition 15** *The* major sequence *of a PED* $\Pi$ *in ND is a sequence of sequents* $\Gamma_1 \vdash A_1, \ldots, \Gamma_n \vdash A_n$ *such that*

1. $\Gamma_1 \vdash A_1$ *is an initial sequent of* $\Pi$,
2. $\Gamma_i \vdash A_i, 1 \leq i < n$, *are major premisses and*
3. $\Gamma_n \vdash A_n$ *is the conclusion of* $\Pi$.

The derivation of figure 4.3 is a PED with the sequents $\Gamma \vdash A \to (B \to C)$, $\Gamma \vdash B \to C$, $\Gamma \vdash C$ forming its major sequence.

**Definition 16** *Let* $\Pi$ *be a PED in ND and let* $\Gamma \vdash B$ *be a sequent that belongs to the major sequence of* $\Pi$ *and is not an initial sequent. A* pseudo-derivation $\Sigma$ *of* $\Pi$ *is the tree obtained from* $\Pi$ *by removing every sequent occurrence above* $\Gamma \vdash B$.

A pseudo-derivation can be seen as an "unfinished" PED. As an example, $\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A}\,Ax \quad \dfrac{}{\Gamma \vdash A \to B}\,Ax}{\Gamma \vdash B}\,E_\to \quad \Gamma \vdash B \to C}{\Gamma \vdash C}\,E_\to$ and $\Gamma \vdash C$ are pseudo-derivations of the derivation of figure 4.3. As the derivation of figure 4.2 is not a PED, there is no pseudo-derivation associated to it.

**Definition 17** *A cut-free derivation* $\Pi$ *in LJT is a* pure left derivation (PLD) *if there exists a sequence of sequents* $\Gamma_1; A_1 \vdash B_1$, $\ldots$, $\Gamma_n; A_n \vdash B_n$, $\Gamma_{n+1}; \vdash B_{n+1}$ *such that*

– $\Gamma_1; A_1 \vdash B_1$ *is either an initial sequent (which means that* $A_1 = B_1$*) or the conclusion of a* $\vee\vdash$*-rule.*

– $\Gamma_i; A_i \vdash B_i$, $1 < i \leq n$, *is the conclusion of the application of the left rule of which* $\Gamma_{i-1}; A_{i-1} \vdash B_{i-1}$ *is a premiss and,*

– $\Gamma_{n+1}; \vdash B_{n+1}$ *is the conclusion of the derivation and it is the only sequent in the sequence that has no formula in the stoup.*

*Note that, in a cut-free derivation, $\Gamma_i; A_i \vdash B_i, 1 < i \leq n$, are premisses of left rules. Such a sequence is called the* major sequence *of* $\Pi$.

In other words, a PLD is a derivation without occurrences of right-rules in its main branch. The derivation of figure 4.4 is a PLD with the sequents $\Gamma; C \vdash C$, $\Gamma; B \to C \vdash C$, $\Gamma; A \to (B \to C) \vdash C$, $\Gamma; \vdash C$ forming its major sequence.

**Definition 18** *Let $\Pi$ be a PLD in LJT and let $\Gamma; A \vdash B$ be a sequent that belongs to the major sequence of $\Pi$. A* pseudo-derivation *$\Sigma$ of $\Pi$ is the tree obtained from $\Pi$ by removing every sequent occurrence below $\Gamma; A \vdash B$.*

A pseudo-derivation can be seen as a PLD where the bottom part is missing. As an example,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; A \vdash A}\,Ax}{\Gamma; \vdash A}\,\mathcal{D} \quad \dfrac{}{\Gamma; B \vdash B}\,Ax}{\Gamma; A \to B \vdash B}\,\to\vdash}{\Gamma; \vdash B}\,\mathcal{D} \quad \dfrac{}{\Gamma; C \vdash C}\,Ax}{\Gamma; B \to C \vdash C}\,\to\vdash$$

and $\dfrac{}{\Gamma; C \vdash C}\,Ax$ are pseudo-derivations of the derivation of figure 4.4. As the derivation of figure 4.1 is not a PLD, there is no pseudo-derivation associated to it.

The notions of sequence, major sequence and length of a pseudo-derivation can be easily derived from the previous definitions.

**Lemma 1** *A cut-free derivation $\Pi$ is a PLD iff the bottommost rule applied in $\Pi$ is $\mathcal{D}$.*

*Proof*: ($\Rightarrow$) Straight from the definition. ($\Leftarrow$) By induction over the structure of $\Pi$. ■

From the previous lemma we infer that, as $\Pi$ is a PLD, the uppermost rule applied in $\Pi$ is also an elimination rule.

**Lemma 2** *A normal derivation $\Pi$ is a PED iff the bottommost rule applied in $\Pi$ is an elimination rule.*

*Proof*: ($\Rightarrow$) Straight from the definition. ($\Leftarrow$) By induction over the structure of $\Pi$. ■

**Lemma 3** *If $\Pi$ is a PED in ND, then, if there is an application $r$ of $E_\vee$ in the major sequence of $\Pi$, then $r$ is the topmost rule applied in $\Pi$.*

*Proof*: If $r$ is not the last rule applied in $\Pi$, then the conclusion of $r$ is a major premiss and by definition 10, $\Pi$ is not normal. Hence, $\Pi$ is not a PED.
■

**Corollary 1** *In the major sequence of a PED in ND, there is at most one application of $E_\vee$.*

**Observation 1** *If $\Pi$ is a PED in ND, then $\Pi$ is either of the form*

$$\begin{array}{c}\dfrac{\dfrac{}{\Gamma \vdash A \wedge B}\,{\scriptstyle Ax}}{\Gamma \vdash A}\,{\scriptstyle E_\wedge}\\ \Sigma'\end{array} \quad or \quad \begin{array}{c}\dfrac{\dfrac{}{\Gamma \vdash B \wedge A}\,{\scriptstyle Ax}}{\Gamma \vdash A}\,{\scriptstyle E_\wedge}\\ \Sigma'\end{array} \quad or \quad \begin{array}{c}\dfrac{\dfrac{}{\Gamma \vdash A \to B}\,{\scriptstyle Ax} \quad \begin{array}{c}\Pi'\\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}\,{\scriptstyle E_\to}\\ \Sigma'\end{array}$$

$$or \quad \begin{array}{c}\dfrac{\dfrac{}{\Gamma \vdash \bot}\,{\scriptstyle Ax}}{\Gamma \vdash A}\,{\scriptstyle E_\bot}\\ \Sigma'\end{array} \quad or \quad \dfrac{\dfrac{}{\Gamma \vdash A \vee B}\,{\scriptstyle Ax} \quad \begin{array}{c}\Pi_1\\ \Gamma, A \vdash D\end{array} \quad \begin{array}{c}\Pi_2\\ \Gamma, B \vdash D\end{array}}{\Gamma \vdash D}\,{\scriptstyle E_\vee}$$

**Lemma 4** *If $\Gamma; A_1 \vdash B_1, \ldots, \Gamma; A_n \vdash B_n, \Gamma; A_{n+1} \vdash B_{n+1}$ is a major sequence, then $B_1 = \cdots = B_n = B_{n+1}$.*

*Proof*: Just note that in any left rule, both the premiss which contains a head-formula and the conclusion have a formula of the same form in the right side of the sequents. ■

## 4.5 Translation between normal derivations

Now that we have both ND and LJT defined, let us see some examples of derivations in both systems.

The only possible cut-free/normal derivations of $A \to (B \to (B \to (A \to B)))$ in LJT and in ND, respectively, are:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B; B \vdash B}\,{\scriptstyle Ax}}{A, B; \vdash B}\,{\scriptstyle \mathcal{D}}}{A, B; \vdash A \to B}\,{\scriptstyle \vdash\to}}{A, B; \vdash B \to (A \to B)}\,{\scriptstyle \vdash\to}}{A; \vdash B \to (B \to (A \to B))}\,{\scriptstyle \vdash\to}}{; \vdash A \to (B \to (B \to (A \to B)))}\,{\scriptstyle \vdash\to} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,{\scriptstyle Ax}}{A, B \vdash A \to B}\,{\scriptstyle I_\to}}{A, B \vdash B \to (A \to B)}\,{\scriptstyle I_\to}}{A \vdash B \to (B \to (A \to B))}\,{\scriptstyle I_\to}}{\vdash A \to (B \to (B \to (A \to B)))}\,{\scriptstyle I_\to}$$

The following derivations examplify a reason to keep a copy of the formula when we apply the rule $\mathcal{D}$. In the LJT derivation, we used the premiss $A \wedge (A \to B)$ twice:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge (A \to B); A \vdash A}\,{\scriptstyle Ax}}{A \wedge (A \to B); A \wedge (A \to B) \vdash A}\,{\scriptstyle \wedge\vdash}}{A \wedge (A \to B); \vdash A}\,{\scriptstyle \mathcal{D}} \quad \dfrac{}{A \wedge (A \to B); B \vdash B}\,{\scriptstyle Ax}}{A \wedge (A \to B); A \to B \vdash B}\,{\scriptstyle \to\vdash}}{A \wedge (A \to B); A \wedge (A \to B) \vdash B}\,{\scriptstyle \wedge\vdash}}{A \wedge (A \to B); \vdash B}\,{\scriptstyle \mathcal{D}}$$

$$\dfrac{\dfrac{\dfrac{}{A\wedge(A\to B)\vdash A\wedge(A\to B)}\,{\scriptstyle Ax}}{A\wedge(A\to B)\vdash A\to B}\,{\scriptstyle E_\wedge}\qquad \dfrac{\dfrac{}{A\wedge(A\to B)\vdash A\wedge(A\to B)}\,{\scriptstyle Ax}}{A\wedge(A\to B)\vdash A}\,{\scriptstyle E_\wedge}}{A\wedge(A\to B)\vdash B}\,{\scriptstyle E_\to}$$

Let us compare the derivations of $C$ from $\Gamma=\{A,\,A\to B,\,A\to(B\to C)\}$ in ND (figure 4.3) and in LJT (figure 4.4), where the bold formulas are the active formulas of the major sequence of the derivations.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\vdash A}\,{\scriptstyle Ax}\quad \dfrac{}{\Gamma\vdash A\to B}\,{\scriptstyle Ax}}{\Gamma\vdash B}\,{\scriptstyle E_\to}\qquad \dfrac{\dfrac{}{\Gamma\vdash A}\,{\scriptstyle Ax}\quad \dfrac{}{\Gamma\vdash \mathbf{A}\to(\mathbf{B}\to\mathbf{C})}\,{\scriptstyle Ax}}{\Gamma\vdash \mathbf{B}\to\mathbf{C}}\,{\scriptstyle E_\to}}{\Gamma\vdash\mathbf{C}}\,{\scriptstyle E_\to}$$

Figure 4.3: Example of a pure elimination derivation in ND

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;A\vdash A}\,{\scriptstyle Ax}}{\Gamma;\vdash A}\,{\scriptstyle \mathcal{D}}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;A\vdash A}\,{\scriptstyle Ax}}{\Gamma;\vdash A}\,{\scriptstyle \mathcal{D}}\quad \dfrac{}{\Gamma;B\vdash B}\,{\scriptstyle Ax}}{\Gamma;A\to B\vdash B}\,{\scriptstyle \to\vdash}}{\Gamma;\vdash B}\,{\scriptstyle \mathcal{D}}\qquad \dfrac{}{\Gamma;\mathbf{C}\vdash C}\,{\scriptstyle Ax}}{\Gamma;\mathbf{B}\to\mathbf{C}\vdash C}\,{\scriptstyle \to\vdash}}{\Gamma;\mathbf{A}\to(\mathbf{B}\to\mathbf{C})\vdash C}\,{\scriptstyle \to\vdash}}{\Gamma;\vdash C}\,{\scriptstyle \mathcal{D}}$$

Figure 4.4: Example of a pure elimination derivation in LJT

There are some things we want to call attention to in this example:

1. The bold formulas belong to sequents that form the major sequence of each derivation.

2. Note that in the ND derivation those formulas are in the right side of the sequents and in the LJT derivation they are in the left side of the sequents, in the stoup.

3. Note also that their "order" is inverted, that is, $A\to(B\to C)$, for instance, is in the head of the sequence in ND, but in the bottom in LJT.

4. Note also the relation between (a) the conclusion of the ND derivation and the initial sequent of the major sequence of the LJT derivation and (b) the initial sequent of the major sequence of the ND derivation and the premiss of the bottommost rule (rule $\mathcal{D}$) applied in the LJT derivation.

5. All the formulas contracted in the LJT derivation form axioms in the ND derivation.

6. In the ND derivation, the premisses of the sequents that belong to the same major sequence are the same. Conversely, in the LJT derivation, the sequents that belong to the same major sequence have conclusions of the same form.

Hence, we need to define a translation that takes head-formulas in LJT to the right side of sequents in ND, formulas that goes to the stoup through the rule $\mathcal{D}$ in LJT to axioms in ND and the bottommost left rule applied in a derivation in LJT to the uppermost elimination rule applied in a derivation in ND.

Before defining the translations, we present two trivial results:

**Lemma 5** *Let* $\begin{array}{c}\Psi_i\\ \Gamma_i;\Delta_i \vdash A_i\end{array}$, $1 \leq i \leq n$, *be (pseudo-)derivations in LJT. If* $\dfrac{\Gamma_1;\Delta_1 \vdash A_1 \quad \Gamma_n;\Delta_n \vdash A_n}{\Gamma;\Delta \vdash A}\,r$, $n = 1, 2$, *is a rule in LJT, then*

1. *If $r$ is a right rule or a rule $\mathcal{D}$ (which means that $\Delta = \emptyset$), then* $\dfrac{\begin{array}{c}\Psi_1\\ \Gamma_1;\Delta_1 \vdash A_1\end{array} \quad \begin{array}{c}\Psi_n\\ \Gamma_n;\Delta_n \vdash A_n\end{array}}{\Gamma;\vdash A}\,r$ *is a derivation in LJT.*

2. *If $r$ is a left rule (which means that $\Delta = \{B\}$), then* $\dfrac{\begin{array}{c}\Psi_1\\ \Gamma_1;\Delta_1 \vdash A_1\end{array} \quad \begin{array}{c}\Psi_n\\ \Gamma_n;\Delta_n \vdash A_n\end{array}}{\Gamma;B \vdash A}\,r$ *is a pseudo-derivation of a derivation in LJT.*

*Proof*: The proof is straight from the definition of derivation and of pseudo-derivation. Note that, as there is a head-formula in the conclusion of $r$ in item 2, $r$ cannot be a right rule. ∎

**Lemma 6** *Let* $\dfrac{\Gamma_1 \vdash A_1 \quad \dots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}\,r$ *be a rule in ND and* $\begin{array}{c}\Pi_i\\ \Gamma_i \vdash A_i\end{array}$, $1 \leq i \leq n$, *be normal derivations in ND.*

1. *If $r$ is an introduction rule, then* $\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \vdash A_1\end{array} \quad \dots \quad \begin{array}{c}\Pi_n\\ \Gamma_n \vdash A_n\end{array}}{\Gamma \vdash A}\,r$ *is a derivation in ND.*

2. *If $r$ is an elimination rule with $\Gamma_n \vdash A_n$ as major premiss and* $\begin{array}{c}\Gamma \vdash A\\ \Sigma\end{array}$ *is a pseudo-derivation of a derivation in ND, then*

   (a) $\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \vdash A_1\end{array} \quad \dots \quad \begin{array}{c}\Pi_n\\ \Gamma_n \vdash A_n\end{array}}{\Gamma \vdash A}\,r\\ \Sigma\end{array}$ *is a derivation in ND.*

*(b)* $\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1 \vdash A_1\end{array} \quad \ldots \quad \begin{array}{c}\Pi_{n-1}\\ \Gamma_{n-1} \vdash A_{n-1}\end{array} \quad \Gamma_n \vdash A_n}{\begin{array}{c}\Gamma \vdash A\\ \Sigma\end{array}}\, r$ *is a pseudo-derivation of a derivation in ND.*

*Proof*: The proof is straight from the definition of pseudo-derivation and of derivation in ND.

■

**Definition 19** **(*f*)** *Let* $\Sigma$ *be a pseudo-derivation of a derivation* $\Pi$ *in LJT. If* $g$ *is a translation from cut-free derivations in LJT to normal derivations in ND, then the translation* $f$ *of pseudo-derivations in LJT to pseudo-derivations in ND is defined recursively as follows:*

*If* $\Sigma = \dfrac{}{\Gamma; C \vdash C}\,Ax$ , *then* $f(\Sigma) = \Gamma \vdash C$.

*If* $\Sigma = \dfrac{\begin{array}{c}\Pi'\\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}\Sigma'\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}\,{\to}{\vdash}$ , *then* $f(\Sigma) = \dfrac{\Gamma \vdash A \quad \begin{array}{c}g(\Pi')\\ \Gamma \vdash A \to B\end{array}}{\begin{array}{c}\Gamma \vdash B\\ f(\Sigma')\end{array}}\,E_{\to}$

*If* $\Sigma = \dfrac{\begin{array}{c}\Sigma'\\ \Gamma; A \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}\,{\wedge}{\vdash}$ , *then* $f(\Sigma) = \dfrac{\Gamma \vdash A \wedge B}{\begin{array}{c}\Gamma \vdash A\\ f(\Sigma')\end{array}}\,E_{\wedge}$

*If* $\Sigma = \dfrac{\begin{array}{c}\Sigma'\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}\,{\wedge}{\vdash}$ , *then* $f(\Sigma) = \dfrac{\Gamma \vdash A \wedge B}{\begin{array}{c}\Gamma \vdash B\\ f(\Sigma')\end{array}}\,E_{\wedge}$

*If* $\Sigma = \dfrac{\begin{array}{c}\Pi_1\\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c}\Pi_2\\ \Gamma, B; \vdash C\end{array}}{\Gamma; A \vee B \vdash C}\,{\vee}{\vdash}$ ,

*then* $f(\Sigma) = \dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c}g(\Pi_1)\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}g(\Pi_2)\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,E_{\vee}$

*If* $\Sigma = \dfrac{\begin{array}{c}\Sigma'\\ \Gamma; A \vdash C\end{array}}{\Gamma; \perp \vdash C}\,{\perp}{\vdash}$ , *then* $f(\Sigma) = \dfrac{\Gamma \vdash \perp}{\begin{array}{c}\Gamma \vdash A\\ f(\Sigma')\end{array}}\,E_{\perp}$

Note the role that $C$ plays on the translation. All the sequents of the major sequence of $\Sigma$ have conclusion $C$, but $C$ "disappears" in the translation. It is so because the active formula of the major sequents in LJT are in the left side of the sequents while the active formulas in the major sequents of ND are on the right side of the sequent. The conclusion $C$ only appears in the conclusion of $g(\Pi)$.

**Lemma 7** *Let $g$ be a translation from cut-free derivations in LJT to normal derivations in ND. If $\Sigma$ is a pseudo-derivation in LJT, then $f(\Sigma)$ is a pseudo-derivation in ND.*

*Proof*: The proof is by induction on the length of $\Sigma$ and follows straight from the definition of $f$ (definition 19). We show one case as an example:

Let $\Sigma = \dfrac{\begin{matrix}\Pi' & \Sigma' \\ \Gamma; \vdash A & \Gamma; B \vdash C\end{matrix}}{\Gamma; A \to B \vdash C}{\scriptstyle \to\vdash}$. By def. 19, $f(\Sigma) = \begin{matrix} \dfrac{\begin{matrix} g(\Pi') & \\ \Gamma \vdash A & \Gamma \vdash A \to B\end{matrix}}{\Gamma \vdash B}{\scriptstyle E_\to} \\ f(\Sigma')\end{matrix}$.

By hypothesis, $g(\Pi')$ is a derivation in ND and by induction hypothesis $f(\Sigma')$ is a pseudo-derivation of a derivation in ND. Hence, by lemma 6, $f(\Sigma)$ is a pseudo-derivation of a derivation in ND.

■

**Definition 20** *Let $\Pi$ be a cut-free derivation of LJT. The translation $g$ of cut-free derivations in LJT to normal derivations in ND is defined recursively as follows:*

**Basic case:** *If* $\Pi = \dfrac{\dfrac{}{\Gamma; A \vdash A}{\scriptstyle Ax}}{\Gamma; \vdash A}{\scriptstyle \mathcal{D}}$ , *then* $g(\Pi) = \dfrac{}{\Gamma, A \vdash A}{\scriptstyle Ax}$

$(\vdash\to)$ *If* $\Pi = \dfrac{\begin{matrix}\Pi' \\ \Gamma, A; \vdash B\end{matrix}}{\Gamma; \vdash A \to B}{\scriptstyle \vdash\to}$ , *then* $g(\Pi) = \dfrac{\begin{matrix}g(\Pi') \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \to B}{\scriptstyle I_\to}$

$(\vdash\wedge)$ *If* $\Pi = \dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ \Gamma; \vdash A & \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \wedge B}{\scriptstyle \vdash\wedge}$ , *then* $g(\Pi) = \dfrac{\begin{matrix}g(\Pi_1) & g(\Pi_2) \\ \Gamma \vdash A & \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \wedge B}{\scriptstyle I_\wedge}$

$(\vdash\vee)$ *If* $\Pi = \dfrac{\begin{matrix}\Pi' \\ \Gamma; \vdash A\end{matrix}}{\Gamma; \vdash A \vee B}{\scriptstyle \vdash\vee}$ , *then* $g(\Pi) = \dfrac{\begin{matrix}g(\Pi') \\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \vee B}{\scriptstyle I_\vee}$

*If* $\Pi = \dfrac{\begin{matrix}\Pi' \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \vee B}{\scriptstyle \vdash\vee}$ , *then* $g(\Pi) = \dfrac{\begin{matrix}g(\Pi') \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \vee B}{\scriptstyle I_\vee}$

$(\mathcal{D})$ *If* $\Pi = \dfrac{\begin{matrix}\Sigma' \\ \Gamma, A; A \vdash B\end{matrix}}{\Gamma, A; \vdash B}{\scriptstyle \mathcal{D}}$ , *then* $g(\Pi) = \begin{matrix}\dfrac{}{\Gamma, A \vdash A}{\scriptstyle Ax} \\ f(\Sigma')\end{matrix}$

**Theorem 3** *If $\Pi$ is a cut-free derivation in LJT, then $g(\Pi)$ is a normal derivation in ND.*

*Proof*: The proof is by induction on the length of $\Pi$ and follows straight from the definition 20 and lemma 7. We show one case as an example:

Let $\Pi = \dfrac{\begin{matrix}\Sigma' \\ \Gamma, A; A \vdash B\end{matrix}}{\Gamma, A; \vdash B}{\scriptstyle \mathcal{D}}$ . By definition 20, $g(\Pi) = \begin{matrix}\dfrac{}{\Gamma, A \vdash A}{\scriptstyle Ax} \\ f(\Sigma')\end{matrix}$ . As every sub-derivation of $\Sigma'$ is smaller than $\Pi$, by induction hypothesis and by lemma 7, $f(\Sigma')$ is a pseudo-derivation of a derivation in ND. Hence, by lemma 6, $g(\Pi)$ is a derivation in ND. ■

**Definition 21** *Let $\Sigma$ be a pseudo-derivation of a PED $\Pi$ in ND. If $t$ is a translation from cut-free derivations in LJT to normal derivations in ND, then the translation $s$ of pseudo-derivations in ND to pseudo-derivations in LJT is defined recursively as follows, where $\Gamma \vdash C$ is the conclusion of $\Pi$:*

1. *If* $\Sigma = \Gamma \vdash C$, *then* $s(\Sigma) = \dfrac{}{\Gamma; C \vdash C}\,Ax$

2. *If* $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,E_\wedge \\ \Sigma'\end{array}$, *then* $s(\Sigma) = \dfrac{\begin{array}{c}s(\Sigma') \\ \Gamma; A \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}\,\wedge\vdash$

3. *If* $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash B \wedge A}{\Gamma \vdash A}\,E_\wedge \\ \Sigma'\end{array}$, *then* $s(\Sigma) = \dfrac{\begin{array}{c}s(\Sigma') \\ \Gamma; A \vdash C\end{array}}{\Gamma; B \wedge A \vdash C}\,\wedge\vdash$

4. *If* $\Sigma = \begin{array}{c}\dfrac{\begin{array}{c}\Pi' \\ \Gamma \vdash A\end{array} \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\,E_\to \\ \Sigma'\end{array}$, *then* $s(\Sigma) = \dfrac{\begin{array}{c}t(\Pi') \\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}s(\Sigma') \\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}\,\to\vdash$

5. *If* $\Sigma = \dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c}\Pi_1 \\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\Pi_2 \\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,E_\vee$,

   *then* $s(\Sigma) = \dfrac{\begin{array}{c}t(\Pi_1) \\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c}t(\Pi_2) \\ \Gamma, B; \vdash C\end{array}}{\Gamma; A \vee B \vdash C}\,\vee\vdash$

6. *If* $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}\,E_\bot \\ \Sigma'\end{array}$, *then* $t(\Sigma) = \dfrac{\begin{array}{c}s(\Sigma') \\ \Gamma; A \vdash C\end{array}}{\Gamma; \bot \vdash C}\,\bot\vdash$

**Lemma 8** *Let $t$ be a translation from normal derivations in ND to cut-free derivations in LJT. If $\Sigma$ is a pseudo-derivation in ND, then $s(\Sigma)$ is a pseudo-derivation in LJT.*

*Proof*: The proof is by induction on the length of $\Sigma$ and follows straight from the definition of $s$ (definition 21). We show one case as an example:

Let $\Sigma = \dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c}\Pi_1 \\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\Pi_2 \\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,E_\vee$. By the definition 21,

$$s(\Sigma) = \dfrac{\begin{array}{c}t(\Pi_1) \\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c}t(\Pi_2) \\ \Gamma, B; \vdash C\end{array}}{\Gamma; A \vee B \vdash C}\,\vee\vdash$$

By hypothesis, both $t(\Pi_1)$ and $t(\Pi_2)$ are derivations in LJT. Hence, by lemma 6, $s(\Sigma)$ is a pseudo-derivation of a derivation in ND. ■

**Definition 22** *Let $\Pi$ be a normal derivation in ND. The translation $t$ of normal derivations in ND to cut-free derivations in LJT is defined recursively as follows, where $\Gamma \vdash C$ is the conclusion of $\Pi$:*

**Basic case:** *If* $\Pi = \dfrac{}{\Gamma \vdash A}\,Ax$ *, then* $t(\Pi) = \dfrac{\dfrac{}{\Gamma; A \vdash A}\,Ax}{\Gamma; \vdash A}\,\mathcal{D}$

$(I_{\to})$ *If* $\Pi = \dfrac{\begin{matrix}\Pi' \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \to B}\,I_{\to}$ *, then* $t(\Pi) = \dfrac{\begin{matrix}t(\Pi') \\ \Gamma, A; \vdash B\end{matrix}}{\Gamma; \vdash A \to B}\,\vdash\to$

$(I_{\wedge})$ *If* $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash A\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \wedge B}\,I_{\wedge}$ *, then* $t(\Pi) = \dfrac{\begin{matrix}t(\Pi_1) \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}t(\Pi_2) \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \wedge B}\,\vdash\wedge$

$(I_{\vee})$ *If* $\Pi = \dfrac{\begin{matrix}\Pi' \\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \vee B}\,I_{\vee}$ *, then* $t(\Pi) = \dfrac{\begin{matrix}t(\Pi') \\ \Gamma; \vdash A\end{matrix}}{\Gamma; \vdash A \vee B}\,\vdash\vee$

*If* $\Pi = \dfrac{\begin{matrix}\Pi' \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \vee B}\,I_{\vee}$ *, then* $t(\Pi) = \dfrac{\begin{matrix}t(\Pi') \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \vee B}\,\vdash\vee$

$(E_{\to}, E_{\wedge}, E_{\vee}, E_{\perp})$ *As the last rule applied in $\Pi$ is an elimination rule then, by lemma 2, $\Pi$ is a PED. Hence, $\Pi$ has one of the forms shown in observation 1 and the translation is as follows:*

*If* $\Pi = \begin{matrix}\dfrac{\dfrac{}{\Gamma \vdash A \wedge B}\,Ax}{\Gamma \vdash A}\,E_{\wedge} \\ \Sigma'\end{matrix}$ *, then* $t(\Pi) = \dfrac{\dfrac{\begin{matrix}s(\Sigma') \\ \Gamma; A \vdash C\end{matrix}}{\Gamma; A \wedge B \vdash C}\,\wedge\vdash}{\Gamma; \vdash C}\,\mathcal{D}$

*If* $\Pi = \begin{matrix}\dfrac{\dfrac{}{\Gamma \vdash B \wedge A}\,Ax}{\Gamma \vdash A}\,E_{\wedge} \\ \Sigma'\end{matrix}$ *, then* $t(\Pi) = \dfrac{\dfrac{\begin{matrix}s(\Sigma') \\ \Gamma; A \vdash C\end{matrix}}{\Gamma; B \wedge A \vdash C}\,\wedge\vdash}{\Gamma; \vdash C}\,\mathcal{D}$

*If* $\Pi = \begin{matrix}\dfrac{\begin{matrix}\Pi' \\ \Gamma \vdash A\end{matrix} \quad \dfrac{}{\Gamma \vdash A \to B}\,Ax}{\Gamma \vdash B}\,E_{\to} \\ \Sigma'\end{matrix}$ *, then* $t(\Pi) = \dfrac{\dfrac{\begin{matrix}t(\Pi') \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}s(\Sigma') \\ \Gamma; B \vdash C\end{matrix}}{\Gamma; A \to B \vdash C}\,\to\vdash}{\Gamma; \vdash C}\,\mathcal{D}$

*If* $\Pi = \dfrac{\dfrac{}{\Gamma \vdash A \vee B}\,Ax \quad \begin{matrix}\Pi_1 \\ \Gamma, A \vdash C\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma, B \vdash C\end{matrix}}{\Gamma \vdash C}\,E_{\vee}$ *,*

*then* $t(\Pi) = \dfrac{\dfrac{\begin{matrix}t(\Pi_1) \\ \Gamma, A; \vdash C\end{matrix} \quad \begin{matrix}t(\Pi_2) \\ \Gamma, B; \vdash C\end{matrix}}{\Gamma; A \vee B \vdash C}\,\vee\vdash}{\Gamma; \vdash C}\,\mathcal{D}$

$$\textit{If } \Pi = \begin{array}{c} \dfrac{\dfrac{}{\Gamma \vdash \bot}\,Ax}{\Gamma \vdash A}\,E_\bot \\ \Sigma' \end{array}, \textit{ then } t(\Pi) = \dfrac{\dfrac{\begin{array}{c} s(\Sigma') \\ \Gamma; A \vdash C \end{array}}{\Gamma; \bot \vdash C}\,\wedge\vdash}{\Gamma; \vdash C}\,\mathcal{D}$$

**Theorem 4** *If $\Pi$ is a normal derivation in ND, then $t(\Pi)$ is a cut-free derivation in LJT.*

*Proof*: The proof is by induction on the length of $\Pi$ and follows straight from definition 22 and lemma 8. We show one case as an example:

Let $\Pi = \begin{array}{c} \dfrac{\begin{array}{c}\Pi' \\ \Gamma \vdash A\end{array} \quad \dfrac{}{\Gamma \vdash A \to B}\,Ax}{\Gamma \vdash B}\,E_\to \\ \Sigma' \end{array}$. By def. 22, $t(\Pi) = \dfrac{\dfrac{\begin{array}{c}t(\Pi') \\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}s(\Sigma') \\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}\,\to\vdash}{\Gamma; \vdash C}\,\mathcal{D}$,

where $\Gamma \vdash C$ is the conclusion of $\Pi$.

As every sub-derivation of $\Sigma'$ is smaller than $\Pi$, by lemma 8, $s(\Sigma')$ is a pseudo-derivation of a derivation in LJT and by induction hypothesis, $t(\Pi')$ is a cut-free derivation in LJT. Hence, by lemma 5, $t(\Pi)$ is a derivation in LJT. ■

**Lemma 9** *If $g(t(\Pi)) = \Pi$, for every normal derivation $\Pi$ in ND, then for every pseudo-derivation $\Sigma$ of $\Pi$*

$$f(s(\Sigma)) = \Sigma.$$

*Proof*: The proof is by induction on the length of $\Sigma$. Let $\Sigma$ be a pseudo-derivation of a derivation $\Pi$ whose conclusion is $\Gamma \vdash C$.

If $\Sigma = \Gamma \vdash C$, then $f(s(\Sigma)) = f\left(\dfrac{}{\Gamma; C \vdash C}\,Ax\right) = \Gamma \vdash C$.

If $\Sigma = \begin{array}{c} \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A\end{array} \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\,E_\to \\ \Sigma_1 \end{array}$, then

$$f\left(s\left(\Sigma\right)\right) = f\left(\dfrac{\begin{array}{c}t(\Pi_1) \\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}s(\Sigma_1) \\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}\,\to\vdash\right) = \begin{array}{c} \dfrac{\begin{array}{c}g(t(\Pi_1)) \\ \Gamma \vdash A\end{array} \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\,E_\to \\ f(s(\Sigma_1)) \end{array}$$

By hypothesis, $g(t(\Pi_1)) = \Pi_1$ and by IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $f(s(\Sigma)) = \Sigma$.

If $\Sigma = \begin{array}{c} \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,E_\wedge \\ \Sigma_1 \end{array}$, then

$$f\left(s\left(\Sigma\right)\right) = f\left(\dfrac{\begin{array}{c}s(\Sigma_1) \\ \Gamma; A \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}\,\wedge\vdash\right) = \begin{array}{c} \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,E_\wedge \\ f(s(\Sigma_1)) \end{array}$$

By IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $f(s(\Sigma)) = \Sigma$.

If $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,E_\wedge \\ \Sigma_1\end{array}$, then

$$f\left(s\left(\Sigma\right)\right) = f\left(\begin{array}{c} s(\Sigma_1) \\ \dfrac{\Gamma ; B \vdash C}{\Gamma ; A \wedge B \vdash C}\,\wedge\vdash \end{array}\right) = \begin{array}{c}\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,E_\wedge \\ f(s(\Sigma_1))\end{array}$$

By IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $f(s(\Sigma)) = \Sigma$.

If $\Sigma = \dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c}\Pi_1 \\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\Pi_2 \\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,E_\vee$, then $f\left(s\left(\Sigma\right)\right) =$

$$f\left(\dfrac{\begin{array}{c} t(\Pi_1) \\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c} t(\Pi_2) \\ \Gamma, B; \vdash C\end{array}}{\Gamma ; A \vee B \vdash C}\,\vee\vdash\right) = \dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c} g(t(\Pi_1)) \\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c} g(t(\Pi_2)) \\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,E_\vee$$

By hypothesis, $g(t(\Pi_1)) = \Pi_1$ and $g(t(\Pi_2)) = \Pi_2$. Hence, $f(s(\Sigma)) = \Sigma$.

If $\Sigma = \begin{array}{c}\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}\,E_\bot \\ \Sigma_1\end{array}$, then $f\left(s\left(\Sigma\right)\right) = g\left(\begin{array}{c} s(\Sigma_1) \\ \dfrac{\Gamma ; A \vdash C}{\Gamma ; \bot \vdash C}\,\wedge\vdash \end{array}\right) = \begin{array}{c}\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}\,E_\wedge \\ f(s(\Sigma_1))\end{array}$

By IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $f(s(\Sigma)) = \Sigma$.

■

**Theorem 5** *For every normal derivation* $\Pi$ *in ND,*

$$g(t(\Pi)) = \Pi.$$

*Proof*: The proof is by induction on the length of $\Pi$. For the basic case, we have that $g\left(t\left(\dfrac{}{\Gamma \vdash C}\,Ax\right)\right) = g\left(\dfrac{\dfrac{}{\Gamma ; C \vdash C}\,Ax}{\Gamma ; \vdash C}\,Cont\right) = \dfrac{}{\Gamma \vdash C}\,Ax$

Let $r$ be the bottommost rule applied in $\Pi$. Then, we have the following cases:

If $\Pi = \begin{array}{c}\Pi_1 \\ \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,I_\to\end{array}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\begin{array}{c} t(\Pi_1) \\ \dfrac{\Gamma, A; \vdash B}{\Gamma ; \vdash A \to B}\,\vdash\to \end{array}\right) = \begin{array}{c} g(t(\Pi_1)) \\ \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,I_\to \end{array}$$

By IH, $g(t(\Pi_1)) = \Pi_1$. Hence, $g(t(\Pi)) = \Pi$.

If $\Pi = \dfrac{\begin{array}{c}\Pi_1 \\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\Pi_2 \\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \wedge B}\,I_\wedge$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\begin{matrix} t(\Pi_1) & t(\Pi_2) \\ \Gamma; \vdash A & \Gamma; \vdash B \end{matrix}}{\Gamma; \vdash A \wedge B}{\scriptstyle \vdash \wedge}\right) =$$

$$\dfrac{\begin{matrix} g(t(\Pi_1)) & g(t(\Pi_2)) \\ \Gamma \vdash A & \Gamma \vdash B \end{matrix}}{\Gamma \vdash A \wedge B}{\scriptstyle I_\wedge}$$

By IH, $g(t(\Pi_1)) = \Pi_1$ and $g(t(\Pi_2)) = \Pi_2$. Hence, $g(t(\Pi)) = \Pi$

If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \vee B}{\scriptstyle I_\vee}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\begin{matrix} t(\Pi_1) \\ \Gamma; \vdash A \end{matrix}}{\Gamma; \vdash A \vee B}{\scriptstyle \vdash \vee}\right) = \dfrac{\begin{matrix} g(t(\Pi_1)) \\ \Gamma \vdash A \end{matrix}}{\Gamma \vdash A \vee B}{\scriptstyle I_\vee}$$

By IH, $g(t(\Pi_1)) = \Pi_1$. Hence, $g(t(\Pi)) = \Pi$.

If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \vee B}{\scriptstyle I_\vee}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\begin{matrix} t(\Pi_1) \\ \Gamma; \vdash B \end{matrix}}{\Gamma; \vdash A \vee B}{\scriptstyle \vdash \vee}\right) = \dfrac{\begin{matrix} g(t(\Pi_1)) \\ \Gamma \vdash B \end{matrix}}{\Gamma \vdash A \vee B}{\scriptstyle I_\vee}$$

By IH, $g(t(\Pi_1)) = \Pi_1$. Hence, $g(t(\Pi)) = \Pi$.

If the last rule applied in $\Pi$ is an elimination rule then, by lemma 2, $\Pi$ is a PED and has one of the forms shown in observation 1. We have the following cases, where $\Gamma \vdash C$ is the conclusion of $\Pi$:

If $\Pi = \begin{matrix}\dfrac{\begin{matrix}\Pi_1 & \\ \Gamma \vdash A & \dfrac{}{\Gamma \vdash A \to B}{\scriptstyle Ax}\end{matrix}}{\Gamma \vdash B}{\scriptstyle E_\to} \\ \Sigma_1\end{matrix}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\dfrac{\begin{matrix} t(\Pi_1) & s(\Sigma_1) \\ \Gamma; \vdash A & \Gamma; B \vdash C \end{matrix}}{\Gamma; A \to B \vdash C}{\scriptstyle \to\vdash}}{\Gamma; \vdash C}{\scriptstyle \mathcal{D}}\right) = \begin{matrix}\dfrac{\begin{matrix} g(t(\Pi_1)) & \\ \Gamma \vdash A & \dfrac{}{\Gamma \vdash A \to B}{\scriptstyle Ax} \end{matrix}}{\Gamma \vdash B}{\scriptstyle E_\to} \\ f(s(\Sigma_1))\end{matrix}$$

By IH, $g(t(\Pi_1)) = \Pi_1$ and by lemma 9 and IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $g(t(\Pi)) = \Pi$.

If $\Pi = \begin{matrix}\dfrac{\dfrac{}{\Gamma \vdash A \wedge B}{\scriptstyle Ax}}{\Gamma \vdash A}{\scriptstyle E_\wedge} \\ \Sigma_1\end{matrix}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\dfrac{\begin{matrix} s(\Sigma_1) \\ \Gamma; A \vdash C \end{matrix}}{\Gamma; A \wedge B \vdash C}{\scriptstyle \wedge\vdash}}{\Gamma; \vdash C}{\scriptstyle \mathcal{D}}\right) = \begin{matrix}\dfrac{\dfrac{}{\Gamma \vdash A \wedge B}{\scriptstyle Ax}}{\Gamma \vdash A}{\scriptstyle E_\wedge} \\ f(s(\Sigma_1))\end{matrix}$$

By lemma 9 and IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $g(t(\Pi)) = \Pi$.

If $\Pi = \dfrac{\dfrac{}{\Gamma \vdash A \wedge B}{\scriptstyle Ax}}{\begin{array}{c}\Gamma \vdash B \\ \Sigma_1\end{array}}{\scriptstyle E_\wedge}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\dfrac{\begin{array}{c}s(\Sigma_1)\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}{\scriptstyle \wedge\vdash}}{\Gamma; \vdash C}{\scriptstyle \mathcal{D}}\right) = \dfrac{\dfrac{}{\Gamma \vdash A \wedge B}{\scriptstyle Ax}}{\begin{array}{c}\Gamma \vdash B \\ f(s(\Sigma_1))\end{array}}{\scriptstyle E_\wedge}$$

By lemma 9 and IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $g(t(\Pi)) = \Pi$.

If $\Pi = \dfrac{\dfrac{}{\Gamma \vdash A \vee B}{\scriptstyle Ax} \quad \begin{array}{c}\Pi_1\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\Pi_2\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}{\scriptstyle E_\vee}$, then $g\left(t\left(\Pi\right)\right) =$

$$g\left(\dfrac{\dfrac{\begin{array}{c}s(\Pi_1)\\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c}s(\Pi_2)\\ \Gamma, B; \vdash C\end{array}}{\Gamma; A \vee B \vdash C}{\scriptstyle \vee\vdash}}{\Gamma; \vdash C}{\scriptstyle \mathcal{D}}\right) = \dfrac{\dfrac{}{\Gamma \vdash A \vee B}{\scriptstyle Ax} \quad \begin{array}{c}g(s(\Pi_1))\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}g(s(\Pi_2))\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}$$

By hypothesis, $g(s(\Pi_1)) = \Pi_1$ and $g(s(\Pi_2)) = \Pi_2$. Hence, $g(t(\Pi)) = \Pi$.

If $\Pi = \dfrac{\dfrac{}{\Gamma \vdash \bot}{\scriptstyle Ax}}{\begin{array}{c}\Gamma \vdash A \\ \Sigma_1\end{array}}{\scriptstyle E_\bot}$, then

$$g\left(t\left(\Pi\right)\right) = g\left(\dfrac{\dfrac{\begin{array}{c}s(\Sigma_1)\\ \Gamma; A \vdash C\end{array}}{\Gamma; \bot \vdash C}{\scriptstyle \wedge\vdash}}{\Gamma; \vdash C}{\scriptstyle \mathcal{D}}\right) = \dfrac{\dfrac{}{\Gamma \vdash \bot}{\scriptstyle Ax}}{\begin{array}{c}\Gamma \vdash A \\ f(s(\Sigma_1))\end{array}}{\scriptstyle E_\wedge}$$

By lemma 9 and IH, $f(s(\Sigma_1)) = \Sigma_1$. Hence, $g(t(\Pi)) = \Pi$.

$\blacksquare$

**Lemma 10** *If* $t(g(\Pi)) = \Pi$, *for every cut-free derivation* $\Pi$ *in LJT, then for every pseudo-derivation* $\Sigma$ *of* $\Pi$,

$$s(f(\Sigma)) = \Sigma.$$

*Proof*: The proof is by induction on the length of $\Sigma$. Let $\Sigma$ be a pseudo-derivation of a cut-free derivation $\Pi$. We have that:

If $\Sigma = \dfrac{}{\Gamma; C \vdash C}{\scriptstyle Ax}$, then $f(s(\Sigma)) = f(\Gamma \vdash C) = \dfrac{}{\Gamma; C \vdash C}{\scriptstyle Ax}$.

If $\Sigma = \dfrac{\begin{array}{c}\Pi_1\\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}\Sigma_1\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}{\scriptstyle \to\vdash}$, then

$$s\left(f\left(\Sigma\right)\right) = s\left(\dfrac{\begin{array}{c}g(\Pi_1)\\ \Gamma \vdash A\end{array} \quad \Gamma \vdash A \to B}{\begin{array}{c}\Gamma \vdash B\\ f(\Sigma_1)\end{array}}{\scriptstyle E_\to}\right) = \dfrac{\begin{array}{c}t(g(\Pi_1))\\ \Gamma; \vdash A\end{array} \quad \begin{array}{c}s(f(\Sigma_1))\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \to B \vdash C}{\scriptstyle \to\vdash}$$

By hypothesis, $t(g(\Pi_1)) = \Pi_1$ and by IH, $s(f(\Sigma_1)) = \Sigma_1$. Hence, $s(f(\Sigma)) = \Sigma$.

If $\Sigma = \dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; B \vdash C\end{array}}{\Gamma; B \wedge A \vdash C}\,{\scriptstyle \wedge\vdash}$, then

$$s\left(f\left(\Sigma\right)\right) = s\left(\begin{array}{c}\dfrac{\Gamma \vdash B \wedge A}{\Gamma \vdash B}\,{\scriptstyle E_\wedge}\\ f(\Sigma_1)\end{array}\right) = \dfrac{\begin{array}{c}s(f(\Sigma_1))\\ \Gamma; B \vdash C\end{array}}{\Gamma; B \wedge A \vdash C}\,{\scriptstyle \wedge\vdash}$$

By IH, $s(f(\Sigma_1)) = \Sigma_1$. Hence, $s(f(\Sigma)) = \Sigma$.

If $\Sigma = \dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}\,{\scriptstyle \wedge\vdash}$, then

$$s\left(f\left(\Sigma\right)\right) = s\left(\begin{array}{c}\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,{\scriptstyle E_\wedge}\\ f(\Sigma_1)\end{array}\right) = \dfrac{\begin{array}{c}s(f(\Sigma_1))\\ \Gamma; B \vdash C\end{array}}{\Gamma; A \wedge B \vdash C}\,{\scriptstyle \wedge\vdash}$$

By IH, $s(f(\Sigma_1)) = \Sigma_1$. Hence, $s(f(\Sigma)) = \Sigma$.

If $\Sigma = \dfrac{\begin{array}{c}\Pi_1\\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c}\Pi_2\\ \Gamma, B; \vdash C\end{array}}{\Gamma; A \vee B \vdash C}\,{\scriptstyle \vee\vdash}$, then $s\left(f\left(\Sigma\right)\right) =$

$$s\left(\dfrac{\Gamma \vdash A \vee B \quad \begin{array}{c}g(\Pi_1)\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}g(\Pi_2)\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\,{\scriptstyle \vee\vdash}\right) = \dfrac{\begin{array}{c}t(g(\Pi_1))\\ \Gamma, A; \vdash C\end{array} \quad \begin{array}{c}t(g(\Pi_2))\\ \Gamma, B; \vdash C\end{array}}{\Gamma; A \vee B \vdash C}\,{\scriptstyle \vee\vdash}$$

By hypothesis, $t(g(\Pi_1)) = \Pi_1$ and $t(g(\Pi_2)) = \Pi_2$. Hence, $s(f(\Sigma)) = \Sigma$.

If $\Sigma = \dfrac{\begin{array}{c}\Sigma_1\\ \Gamma; A \vdash C\end{array}}{\Gamma; \bot \vdash C}\,{\scriptstyle \bot\vdash}$, then

$$s\left(f\left(\Sigma\right)\right) = s\left(\begin{array}{c}\dfrac{\Gamma; \vdash \bot}{\Gamma; \vdash A}\,{\scriptstyle E_\bot}\\ f(\Sigma_1)\end{array}\right) = \dfrac{\begin{array}{c}s(f(\Sigma_1))\\ \Gamma; A \vdash C\end{array}}{\Gamma; \bot \vdash C}\,{\scriptstyle \bot\vdash}$$

By IH, $s(f(\Sigma_1)) = \Sigma_1$. Hence, $s(f(\Sigma)) = \Sigma$.

■

**Theorem 6** *For every cut-free derivation* $\Pi$ *in LJT,*

$$t(g(\Pi)) = \Pi.$$

*Proof*: Let $\Pi$ be a cut-free derivation in LJT. The proof is by induction on the length of $\Pi$. We have the following cases:

If $\Pi = \dfrac{\dfrac{}{\Gamma; A \vdash A}\,{\scriptstyle Ax}}{\Gamma; \vdash A}\,{\scriptstyle Cont}$, then $t\left(g\left(\Pi\right)\right) = t\left(\dfrac{}{\Gamma \vdash A}\,{\scriptstyle Ax}\right) = \dfrac{\dfrac{}{\Gamma; A \vdash A}\,{\scriptstyle Ax}}{\Gamma; \vdash A}\,{\scriptstyle Cont}$

If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma, A; \vdash B\end{matrix}}{\Gamma; \vdash A \to B} \vdash\to$, then

$$t\left(g\left(\Pi\right)\right) = t\left(\dfrac{\begin{matrix}g(\Pi_1) \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \to B} I_\to\right) = \dfrac{\begin{matrix}t(g(\Pi_1)) \\ \Gamma, A; \vdash B\end{matrix}}{\Gamma; \vdash A \to B} \vdash\to$$

By IH, $t(g(\Pi_1)) = \Pi_1$. Hence, $t(g(\Pi)) = \Pi$.

If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \wedge B} \vdash\wedge$, then

$$t\left(g\left(\Pi\right)\right) = t\left(\dfrac{\begin{matrix}g(\Pi_1) \\ \Gamma \vdash A\end{matrix} \quad \begin{matrix}g(\Pi_2) \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \wedge B} I_\wedge\right) = \dfrac{\begin{matrix}t(g(\Pi_1)) \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}t(g(\Pi_2)) \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \wedge B} \vdash\wedge$$

By IH, $t(g(\Pi_1)) = \Pi_1$ and $t(g(\Pi_2)) = \Pi_2$. Hence, $t(g(\Pi)) = \Pi$.

If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma; \vdash A\end{matrix}}{\Gamma; \vdash A \vee B} \vdash\vee$, then

$$t\left(g\left(\Pi\right)\right) = t\left(\dfrac{\begin{matrix}g(\Pi_1) \\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash A \vee B} I_\vee\right) = \dfrac{\begin{matrix}t(g(\Pi_1)) \\ \Gamma; \vdash A\end{matrix}}{\Gamma; \vdash A \vee B} \vdash\vee$$

By IH, $t(g(\Pi_1)) = \Pi_1$. Hence, $t(g(\Pi)) = \Pi$.

If $\Pi = \dfrac{\begin{matrix}\Pi_1 \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \vee B} \vdash\vee$, then

$$t\left(g\left(\Pi\right)\right) = t\left(\dfrac{\begin{matrix}g(\Pi_1) \\ \Gamma \vdash B\end{matrix}}{\Gamma \vdash A \vee B} I_\vee\right) = \dfrac{\begin{matrix}t(g(\Pi_1)) \\ \Gamma; \vdash B\end{matrix}}{\Gamma; \vdash A \vee B} \vdash\vee$$

By IH, $t(g(\Pi_1)) = \Pi_1$. Hence, $t(g(\Pi)) = \Pi$.

If $\Pi = \dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}\Sigma_1 \\ \Gamma; B \vdash C\end{matrix}}{\Gamma; A \to B \vdash C} \to\vdash}{\Gamma; \vdash C} \mathcal{D}$, then

$$t\left(g\left(\Pi\right)\right) = t\left(\begin{matrix}\dfrac{\begin{matrix}g(\Pi_1) \\ \Gamma \vdash A\end{matrix} \quad \dfrac{}{\Gamma \vdash A \to B} Ax}{\Gamma \vdash B} E_\to \\ f(\Sigma_1)\end{matrix}\right) = \dfrac{\dfrac{\begin{matrix}t(g(\Pi_1)) \\ \Gamma; \vdash A\end{matrix} \quad \begin{matrix}s(f(\Sigma_1)) \\ \Gamma; B \vdash C\end{matrix}}{\Gamma; A \to B \vdash C} \to\vdash}{\Gamma; \vdash C} \mathcal{D}$$

By IH, $t(g(\Pi_1)) = \Pi_1$ and by lemma 10 and IH, $s(f(\Sigma_1)) = \Sigma_1$. Hence, $t(g(\Pi)) = \Pi$.

If $\Pi = \dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ \Gamma; B \vdash C\end{matrix}}{\Gamma; A \wedge B \vdash C} \wedge\vdash}{\Gamma; \vdash C} \mathcal{D}$, then

$$t\left(g\left(\Pi\right)\right)=t\left(\begin{array}{c}\dfrac{\overline{\Gamma\vdash A\wedge B}\;{\scriptstyle Ax}}{\Gamma\vdash B}\;{\scriptstyle E_\wedge}\\ f(\Sigma_1)\end{array}\right)=\dfrac{\dfrac{\begin{array}{c}s(f(\Sigma_1))\\ \Gamma;B\vdash C\end{array}}{\Gamma;A\wedge B\vdash C}\;{\scriptstyle \wedge\vdash}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$$

By lemma 10 and IH, $s(f(\Sigma_1))=\Sigma_1$. Hence, $t(g(\Pi))=\Pi$.

If $\Pi=\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma;B\vdash C\end{array}}{\Gamma;B\wedge A\vdash C}\;{\scriptstyle \wedge\vdash}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$, then

$$t\left(g\left(\Pi\right)\right)=t\left(\begin{array}{c}\dfrac{\overline{\Gamma\vdash B\wedge A}\;{\scriptstyle Ax}}{\Gamma\vdash B}\;{\scriptstyle E_\wedge}\\ f(\Sigma_1)\end{array}\right)=\dfrac{\dfrac{\begin{array}{c}s(f(\Sigma_1))\\ \Gamma;B\vdash C\end{array}}{\Gamma;B\wedge A\vdash C}\;{\scriptstyle \wedge\vdash}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$$

By lemma 10 and IH, $s(f(\Sigma_1))=\Sigma_1$. Hence, $t(g(\Pi))=\Pi$.

If $\Pi=\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma,A;\vdash C\end{array}\quad\begin{array}{c}\Pi_2\\ \Gamma,B;\vdash C\end{array}}{\Gamma;A\vee B\vdash C}\;{\scriptstyle \vee\vdash}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$, then $t\left(g\left(\Pi\right)\right)=$

$$t\left(\dfrac{\overline{\Gamma\vdash A\vee B}\;{\scriptstyle Ax}\quad\begin{array}{c}g(\Pi_1)\\ \Gamma,A\vdash C\end{array}\quad\begin{array}{c}g(\Pi_2)\\ \Gamma,B\vdash C\end{array}}{\Gamma\vdash C}\;{\scriptstyle \vee\vdash}\right)=\dfrac{\dfrac{\begin{array}{c}t(g(\Pi_1))\\ \Gamma,A;\vdash C\end{array}\quad\begin{array}{c}t(g(\Pi_2))\\ \Gamma,B;\vdash C\end{array}}{\Gamma;A\vee B\vdash C}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$$

By IH, $t(g(\Pi_1))=\Pi_1$ and $t(g(\Pi_2))=\Pi_2$. Hence, $t(g(\Pi))=\Pi$.

If $\Pi=\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ \Gamma;A\vdash C\end{array}}{\Gamma;\bot\vdash C}\;{\scriptstyle \bot\vdash}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$, then

$$t\left(g\left(\Pi\right)\right)=t\left(\begin{array}{c}\dfrac{\overline{\Gamma\vdash\bot}\;{\scriptstyle Ax}}{\Gamma\vdash A}\;{\scriptstyle E_\bot}\\ f(\Sigma_1)\end{array}\right)=\dfrac{\dfrac{\begin{array}{c}s(f(\Sigma_1))\\ \Gamma;A\vdash C\end{array}}{\Gamma;\bot\vdash C}\;{\scriptstyle \bot\vdash}}{\Gamma;\vdash C}\;{\scriptstyle \mathcal{D}}$$

By lemma 10 and IH, $s(f(\Sigma_1))=\Sigma_1$. Hence, $t(g(\Pi))=\Pi$.

■

From theorems 5 and 6, we have that the translations defined between ND and LJT are bijective.

## 4.6 Conclusion

We achieved a bijection between normal derivations of Natural Deduction and cut-free derivations of Sequent Calculus. In order to complete the proof-theoretical isomorphism between the systems we have to show that the reduction steps are pair related. In order to do this, the translations $(f, g)$ and $(s, t)$ must be extended to translate any derivation, and not just normal and cut-free ones. Finally, we need to show that the extended translations preserve reductions up to equivalent derivations, which enable us to define a translation between conversions. This is done in next chapter.