Pontifícia Universidade Católica
do Rio de Janeiro

Luciana Schmid Blatter Moreira

Risk Analysis in a Portfolio of Commodities: A Case Study

DISSERTAÇÃO DE MESTRADO

Dissertation presented to the Programa de Pós-Graduação em Engenharia Elétrica of the Departamento de Engenharia Elétrica, PUC-Rio as partial fulfilment of the requirements for the degree of Mestre em Engenharia Elétrica.

Advisor: Prof. Cristiano Augusto Coelho Fernandes

Co-Advisor: Prof. Jorge Passamani Zubelli

Rio de Janeiro

June 2014

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Bibliographic data

Moreira, Luciana Schmid Blatter

Risk Analysis in a Portfolio of Commodities: A Case Study / Luciana Schmid Blatter Moreira; advisor: Cristiano Augusto Coelho Fernandes; co-advisor: Jorge Passamani Zubelli – 2014.

86 f. ; 30 cm

Dissertação (mestrado) – Pontifícia Universidade Católica do Rio de Janeiro, 2014.

Inclui bibliografia

1. Engenharia elétrica – Teses. 2. Principal Component Analysis. 3. Value-at-Risk (VaR). 4. Conditional Value-at-Risk (CVaR). 5. Omega Ratio. 6. Backtesting techniques. 7. Optimization. 8. GARCH. 9. Geometric Brownian Motion. I. Fernandes, Cristiano Augusto Coelho. II. Zubelli, Jorge Passamani. III. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Engenharia Elétrica. IV. Título.

CDD: 621.3

Acknowledgments

Thank God for the wisdom and perseverance that he has been bestowed upon me during this research project, and indeed, throughout my life.

I would like to express my deepest gratitude to my advisor, Dr. Jorge Zubelli, for his excellent guidance, caring, patience, and for teaching me not only mathematics but also lifetime lessons that I will never forget.

I hereby show my greatest appreciation to my advisor, Dr. Cristiano Fernandes, who contributed to my development since under graduation. Your patience, dedication, advices and comments were a great help in the accomplish of this dissertation. And to Betina Fernandes for the relevant comments and corrections.

I would like to thank PUC-Rio, CAPES, Petrobras and IMPA for the financial support during my Master's work.

Last but not least, I would like to thank my parents for their unconditional support, love and encouragement. In particular, the patience, love and understanding shown by my dearest mom, and my dad, for being the greatest example of strength that I ever seen and also, for being my biggest encourager to accomplish this dissertation. I dedicate this dissertation to you, my two greatest love.

A special thanks to my grandmother Maria Enecy. Words cannot express how grateful I am for having you as a grandma. Your prayers for me was what sustained me thus far.

To my sisters Luiza and Ana Carolina that despite the distance, were able to encourage me to never give up. I love you with all my heart.

To my sister and best friend Mariana, for always believing in me and understanding me even in tough times. Your emotional support and daily presence were crucial. You are a huge part of this victory. Thank you for everything.

Finally, I would like also to thank Bruno Fanzeres for the love and dedication. Thank you, my dear, for always cheering me up even in the difficult periods. Our sleepless nights during our Master's course made us stronger. I couldn't have made without you.

To my great friend João Carlos Reis, for the daily talks, instigating discussions and constant encouragement. Our uncountable lunches were essential to keep me motivated.

To my dearest friends, who understood my absence in this last few years. Also, I would also like to thank every LAMPS and LAMCA members for their insightful considerations and for providing a great work atmosphere.

Abstract

Moreira, Luciana Schmid Blatter; Fernandes, Cristiano Augusto Coelho (Advisor); Zubelli, Jorge Passamani (Co-advisor). **Risk Analysis in a Portfolio of Commodities: A Case Study**. Rio de Janeiro, 2014. 86p. MSc Dissertation – Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

One of the main challenges in the financial market is to simulate prices keeping the correlation structure among numerous assets. *Principal Component Analysis* emerges as solution to the latter problem. Also, given the uncertainty present in *commodities markets*, an investor wants to protect his/her assets from potential losses, so as an alternative, the optimization of various risk measures, such as *Value-at-risk*, *Conditional Value-at-risk* and *Omega Ratio*, are important financial tools. Additionally, the backtest is widely used to validate and analyze the performance of the proposed methodology. In this dissertation, we will work with a portfolio of oil commodities. We will put together different techniques and propose a new methodology that consists in the (potentially) decrease the dimension of the proposed portfolio. The following step is to simulate the prices of the assets in the portfolio and then optimize the allocation of the portfolio of oil commodities. Finally, we will use backtest techniques in order to validate our method.

Keywords

Principal Component Analysis; Value-at-Risk (VaR); Conditional Value-at-Risk (CVaR); Omega Ratio; Backtesting Techniques; Optimization; GARCH; Geometric Brownian Motion.

Resumo

Moreira, Luciana Schmid Blatter; Fernandes, Cristiano Augusto Coelho (Orientador); Zubelli, Jorge Passamani (Co-orientador). **Análise de Riscos num Portfólio de Commodities: Um Estudo de Caso.** Rio de Janeiro, 2014. 86p. Dissertação de Mestrado – Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

Um dos principais desafios no mercado financeiro é simular preços mantendo a estrutura de correlação entre os inúmeros ativos de um portfólio. Análise de Componentes Principais emerge como uma solução para este último problema. Além disso, dada a incerteza presente nos mercados de commodities de derivados de petróleo, o investidor quer proteger seus ativos de perdas potenciais. Como uma alternativa a esse problema, a otimização de várias medidas de risco, como Value-at-risk, Conditional Value-at-risk e medida Ômega, são ferramentas financeiras importantes. Além disso, o backtest é amplamente utilizado para validar e analisar o desempenho do método proposto. Nesta dissertação, trabalharemos com um portfólio de commodities de petróleo. Vamos unir diferentes técnicas e propor uma nova metodologia que consiste na diminuição da dimensão do portfólio proposto. O passo seguinte é simular os preços dos ativos na carteira e, em seguida, otimizar a alocação do portfólio de commodities de derivados do petróleo. Finalmente, vamos usar técnicas de backtest, a fim de validar nosso método.

Palavras-chave

Análise de Componentes Principais; Value-at-Risk (VaR); Conditional Value-at-Risk (VaR); Medida Omega; Backtesting Techniques; Otimização; GARCH; Movimento Geometrico Browniano.

“Faber est quisque fortunae suae”

Sallustio

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1
Introduction

Investment is usually defined as the commitment of wealth in order to achieve the investor's objective. Therefore, investment science is the application of scientific tools to investments. These scientific tools used are primarily mathematical and statistical and are required in the *risk analysis* study.

Risk, in Finance, by definition, is the probability of losing something. In Finance, the investor wishes to mitigate or take certain controlled risks. Risk analysis has a wide range of applications, such as: the uncertainty of the portfolio/stock returns, statistical analysis to determine the probability of a given scenario, and possible future economic states.

Investment analysis is the process of examining outcomes on investment and deciding which one is most adequate for the investor, according to his profile. Markowitz [20] was the pioneer in demonstrating the problem of the portfolio selection and the risk mitigation. Besides structuring the mathematical portfolio problem, [20] showed that each investor has a risk profile and tolerates a certain amount of risk in their investments.

A typical approach to measure these risks are the so-called risk measures: *Value-at-Risk* and the *Conditional Value-at-Risk*, being some of the available measures. The first one is defined as a certain quantile (usually 95%) of the returns or prices distribution. The probability level is chosen deep enough in the left tail of the loss distribution to be relevant for the risk decision, but not so deep as to be difficult to estimate with accuracy. This probability level represents the worst-case scenario at the chosen quantile. The problem is that it still remains 5% (if the probability level chosen is 95%) of the possible losses. In face of this problem, the *Conditional Value-at-Risk* emerged as an alternative to the use of the *Value-at-Risk*. By definition, the *Conditional Value-at-Risk* is the average of $(1-\alpha)$ worst-case scenarios.

In order to conciliate risk and optimization, [24] emerged as solution. The authors show the Linear Programming of the *Conditional Value-at-Risk*. In [17], the authors apply the theory of [24] to the benchmark *S&P* using some financial objectives and constrains. In our work, we will present the application of [24] in a very specific portfolio of commodities adding useful financial constrains.

And finally, one should quote *Performance Measures* that are used as complementary to *Risk Measures*. The Sharpe ratio, for example, was the pioneer and is capable to calculate whether a portfolio's returns are due to smart investment decisions or due to a result of excess risk. This kind of measurement is useful because although the portfolio can present higher returns, it is important to analyze if those higher returns do not represent too much risk.

Another valuable and more recent *Performance Measure* is the *Omega Ratio*. The difference between the Sharpe Ratio and the Omega Ratio, is that the latter contains much more information about the return distribution, including the mean, variance, skew and kurtosis and is especially valuable for investments that have non-normal distribution.

The Omega Ratio has several important characteristics that can be valued when comparing portfolios with the same predicted return. Investors should favor the portfolio with the highest Omega Ratio, because this decision maximizes the potential for making the desired level of return, and minimizes the probability of extreme losses. Besides this fact, the entire return distribution, including higher moments, is intrinsic in the Omega ratio. Finally, one can say that the Omega Ratio does not minimize volatility, but it sure reduces the probability of extreme losses. In [13], the authors show the optimization of the Omega measure. In our work, we will present the Linear Programming of the Omega measure adding some financial constraints and also, we will demonstrate case studies using a portfolio of commodities.

If the objective is to mitigate risk, one of the manager's strategies is to provide portfolio diversification. But, by providing the diversification, the number of assets in the portfolio increases. In spite of this fact, the risk analysis and estimation of the portfolio becomes a very complex problem. This discussion can be found in [9].

In order to handle this difficulty, various methods were proposed in the financial literature, such as the *Principal Component Analysis* (PCA) [8], which will be presented in the following chapters of this dissertation. The PCA transforms the series into a different coordinate system, extracting the correlation among the assets of the portfolio. Also, the result of PCA presents the most relevant components that explain the most part of the portfolio. In this way, PCA can (potentially) decrease the portfolio dimensionality. [26] applies this methodology for a specific Market and Shipping. In our work, PCA will be applied to a portfolio of commodities in order to potentially decrease its dimension.

Another instrument that is commonly used in the financial analysis is the *out-*

of-sample test, which provides the behavior of the portfolio when compared to a risk measure, such as *Value-at-Risk*, *Conditional Value-at-Risk* or *Omega Ratio*. The use of the out-of-sample inspires more confidence in the model chosen. In the recent literature, [2] applies this technique in only two assets. In our work, we will apply this technique in a portfolio of 12 assets.

In this work, we will put together these techniques described above. Initially, we will work with a portfolio of 12 assets and we will apply the *Principal Component Analysis* to (potentially) decrease its dimension. Once decided the most relevant components, we will model the scores individually and simulate the prices by keeping the correlation among the assets. The next step is to optimize the *Value-at-Risk*, the *Conditional Value-at-Risk* and the *Omega Ratio* in order to obtain the optimal allocations of the assets that compose our portfolio. Finally, we will validate our methodology by applying a *Backtesting technique*.

1.1 Objectives and Contributions

Presently, one of main challenges of a *portfolio manager* is how to handle the risk of a portfolio with a wide range of assets. The objective of this dissertation is to construct a risk analysis of a portfolio of various assets, including equities and commodities.

Our aim with this dissertation is to:

- Implement a calibration for a multidimensional model taking into account the presence of commodities;
- Perform a dimension reduction of the uncertainty factors associated with the model;
- Implement a portfolio optimization based on the chosen risk measures and the corresponding backtesting.

1.2 Organization

This dissertation is organized as follows: Chapter 2 presents some aspects of the Oil Market, such as: forward oil markets, options, swaps and some *Commodities* estimation models. Chapter 3 presents the most common risk and performance measures and, their linear programming as well. Chapter 4 illustrates the theory of the *Principal Component Analysis*. Chapter 5 presents the backtest theory concerning the violation series. Chapter 6 explains the method used in this work. Chapter 7 presents some case studies to illustrate the accuracy of the proposed methodology. Chapter 8 concludes this dissertation and discusses extensions and future research.

2
Oil as a Commodity and its Particularities

In this chapter, we will present the history and the structure of the *oil market* and the different kinds of financial instruments that can be utilized in this specific market. We will also briefly look at the most popular models to forecast the oil spot prices: the Gibson- Schwartz model [23] and the Schwartz-Smith model [25].

The specificity of the oil market leads to a multitude of contracts related to uncertainties in delivery times, places and types of the commodity. Due to this fact, it is relevant to understand the whole context of commodity trading.

Crude oil is a key natural resource which operates as the underlying asset to many financial instruments, such as futures and options. Physical oil trade can be done in two different ways, by term supply contracts or spot supply contracts. The first one can be defined as a contract for which the seller of oil agrees supplying the buyer a specific quantity of oil at a specific date in the future. It is noteworthy that the majority of oil traded physically is done by term supply contracts. The second type of contract can be defined as a contract for delivery of a specific quantity of oil at a specific location as soon as it is operationally possible. In general, it is done within a day or two. It is relevant to say that the prices at which the spot supply transactions arise will impact to the minute oil market news and any short term changes to supply and demand. This happens because the barrels are sold to the highest bidder.

According to [10], until the 1980s, oil prices in term supply contracts were fixed by the largest oil companies unilaterally or in direct negotiations. Although, fixed contract pricing is still used in some parts of the oil industry, almost all pricing for term supply contracts have changed to benchmark pricing. The latter refers to the spot supply prices. At the end of the day, a snapshot is taken of the supply trading in oil prices of several benchmark grades of oil. The daily prices of oil can be obtained from two sources: *future exchanges* and *trade journals*.

There are two main future exchanges where oil is traded: the New York Mercantile Exchange (NYMEX) and the InterContinental Exchange (ICE), the latter based in London. The trading of oil is also done in cities such as Tokyo, Shanghai and Dubai.

In the trade journal, oil is traded in the Over-The-Counter (OTC) market, which is not physically based anywhere. The most used trade journal for OTC benchmark in oil markets is the so-called *Platts Oilgram*. Platts is a platform that publishes the high and low prices for various grades of oil at the end of every business day. Traders, in general, use the MOP (Mean of Platts) prices as their benchmarks.

Some NYMEX and ICE futures and OTC trade journal grades of oil have become spot market benchmarks. Non-benchmarks grades of oil, such as the price in a specific local area and country are set at a premium or a discount to these benchmark grades. The premium or discount is determined by transportation costs, taxes and quality differences between the grade of oil in a particular area and the benchmark grade.

2.1 Benchmark in Oil pricing

It is well-known that oil is traded globally in US dollars, since it is the most freely convertible and liquid currency with the lowest transaction cost. Besides those advantages, trading oil in a single currency makes it easier to compare oil prices internationally.

One or several benchmarks can be referenced in the calculation of supply contracts, this fact is known as oil formula pricing. Besides the benchmark, the premium or discount of the Official Selling Price (OSP) to the benchmark(s) will evaluate the quality difference between the benchmark crude and the crude grade being priced. The relative demand for the crude being priced has to be considered in the process.

There is a method to determine prices in obscure locations where little trading in physical oil is done. This is the so-called freight netback pricing. It is a variation of benchmark formula pricing, in which the price for a grade of oil at one location is linked to a benchmark oil price at another location. It is adjusted for a benchmark freight cost published by the Baltic Exchange, Platts at some trade journal.

In the US, some crude oil types are traded based on posted prices. Posted prices can be defined as bid prices from refineries and others interested in buying crude oil. In this way, posted prices can be understood as a starting point for negotiating a market price. Often premia are paid above posted prices, known as posting-plus prices, but most posted prices are correlated to benchmark pricing. It is relevant to say that posted prices themselves are not used as benchmark prices.

When physical oil prices are quoted, they are related to a certain point in the delivery chain. The cost of the oil is basically built by crude oil accounts. However, refining and transportation chain add a cost to the oil. In many places, the retail oil prices are different due to the government taxation.

2.2 Foward Oil Markets: Futures and Swaps

A forward contract is defined as the price of oil delivered at a specified date in the future. Usually this trade is done using exchange traded future contracts and Over-The-Counter (OTC) swap contracts. An interesting feature of futures and swaps is that one does not have to be involved in the delivery of the physical oil. Those contracts are the so-called paper barrels, which are known as derivative instruments in financial mathematics.

In order to handle the implicit uncertainty present in the commodities prices, there were created two types of forward curves: *Contango* and *Backwardation*. The difference between them is that a forward curve in contango implies that oil as an asset that has a negative yield for those owning oil (owner of oil has to pay storage costs to continually own the oil) and the futures price is above the expected future spot price, see Figure 2.1.

The most common forward curve is the backwardation, which implies that oil has a positive yield (owner of oil can collect money by selling oil today and buying it back cheaper in the future) and the futures price is below the expected future spot price, as we can see in Figure 2.1.

Presently, changes in spreads due to seasonal differences and economic have not been relevant in absolute prices of crude oil. This can be explained by the spare capacity globally in oil storage, refining and transportation. One can conclude that oil spot prices and forward curves are highly positively correlated. It is well accepted that forward curves do not provide reliable predictions of future spot prices.

It is relevant to remark that a spread position is the exposure of different prices between, for example, two grades of oil, or the same grade of oil over different time periods, or either the same grade of oil in two locations. There are four common oil market spreads: crack, arbitrage, relative value and time spreads.

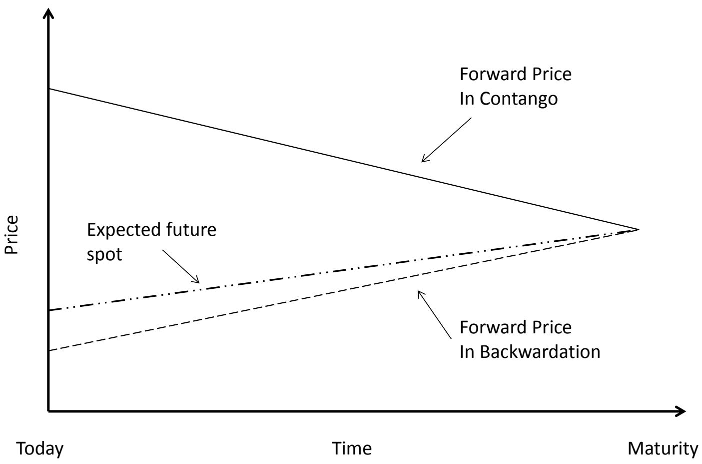

Fig. 2.1: Contango and Backwardation curves

2.3 Forward Oil Markets: Options

Options can be understood as a hedge against the erratic movements of oil prices. The formal definition of option is the right but not the obligation to buy or sell oil (or any asset) at a set price in the future (maturity). The main difference between *options* and *swaps* or *exchange traded futures* is that in swaps contracts, there is the obligation of buying or selling oil at a set price at an certain time in the future, whereas, for options there is no obligation.

A buyer of an option can lose only the premium paid to the option's seller. On the other hand, the seller keeps the premium but has unlimited losses. Normally, there are two options depending on market direction: call options (profit from rising prices), and put options (profit from falling prices). Its payoff are presented in Figures 2.2 and 2.3.

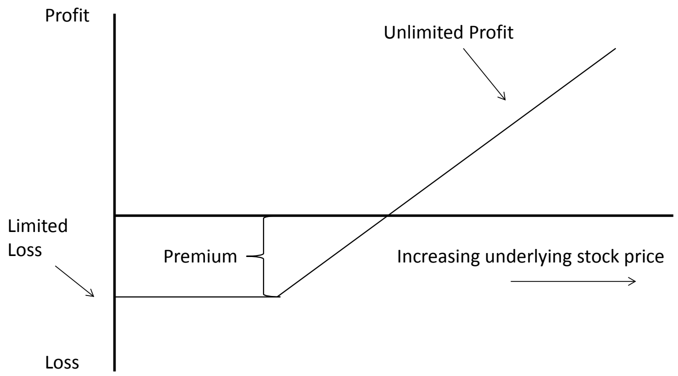

Fig. 2.2: Example of a Call option

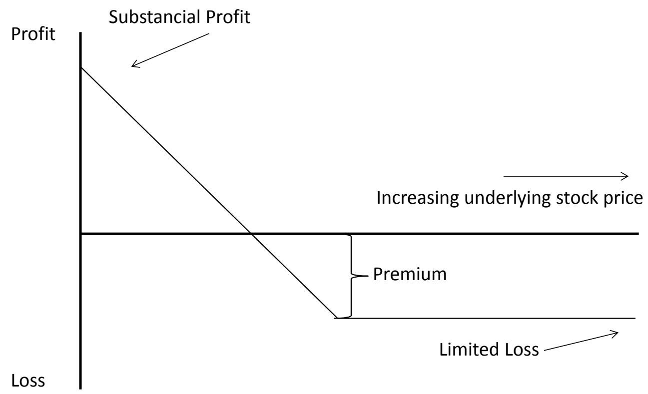

Fig. 2.3: Example of a Put option

Straightforward put and call options, such as those described in Figures 2.2 and 2.3, are known as *plain vanilla option* since they are the most common and simple structures.

The value of an option changes if the underlying market price moves, or if the option gets closer to the expiration day, if interest rates change, or if market volatility change.

Most of options expire in one of the following four ways: American, European, Asian and Bermudan style. It is noteworthy that those names carry no geographical significance and are simply market shorthand for the expiry type. American options are those that can be exercised at any time from the option purchase date up and including its expiry date. Such options are the most expensive ones between these four styles presented.

On the other hand, the Asian options are those, by definition, that can be exercised only at expiry against the arithmetic average price over a period of time, such as a month, a quarter or a year. In European expiry (bullet expiry), the buyer can only exercise against a single price at the end of a period of time. The latter are rarely used in the oil market because they are more expensive than Asian options. Finally, Bermudan expiry can be described as the option that can only be exercised during a defined time window.

It is interesting to say that the Asian style options are more common than American ones in the OTC markets, since oil consumers and producers usually consume and produce oil every day, during a period of time. Also the use of options which match this constant every day consumption and production pattern is more appropriate than American options which expire against a single day's price. Another reason is that by using an average of number of prices for different days (Asian options), the volatility which is used to price the options is lower and consequently, Asian options are cheaper than American ones.

2.4 Managing Oil Price Risk

It is well-known that commodities are one of the most volatile asset class when compared to equities, currencies and bonds. This happens due to fact that commodities are expensive to store. Commodity shortages take months or years to alleviate, whereas equities, currencies, and bonds shortages can quickly be erased by companies issuing more shares, governments printing more currency, and debt issuers borrowing more.

According to Figure 2.4, one can see that hedging involves entering into a transaction to smooth the short term impact of volatile oil price movements. If one looks at the cash flow resulting from hedging over the long term hedging will likely lose a small amount of money as the hedger has to cross a market bid/offer spread. However, the two main reasons to hedge are to reduce short-term cash flow volatility and to maximize return on capital for a target level of risk.

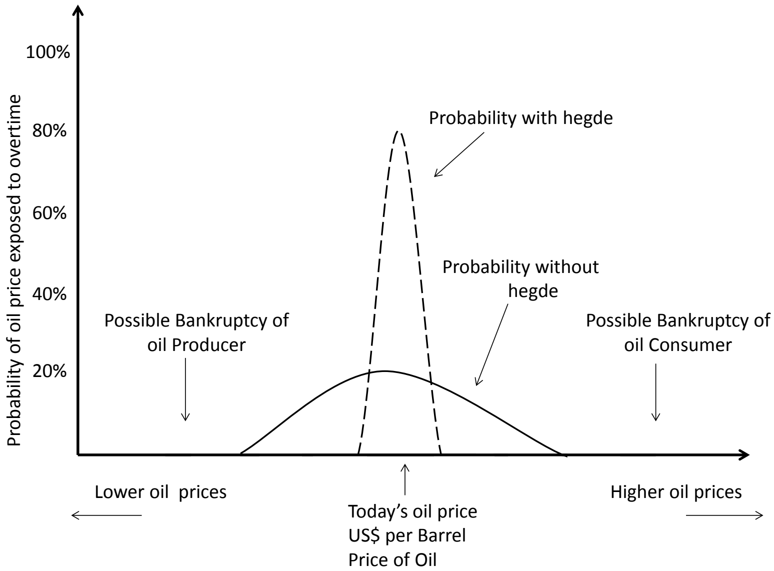

Fig. 2.4: Example of Hedging

First, reducing the volatility of cash flows has many benefits. One of the main ones is the reduction of the risk of bankruptcy, which improves the ability to borrow money and reduces the cost of such borrowing. The second reason for hedging is based on portfolio theory, which is often used to optimize an investment portfolio containing equities, commodities, bonds, cash and other assets, so that the portfolio generates the highest return for the level of risk an investor is comfortable with. As part of its portfolio of costs and revenue, an organization's management can raise or lower the impact of oil price volatility by hedging.

In the next section, we will present the pioneer methodology of forecasting spot price of a certain commodity: The Gibson- Schwartz Model.

2.5 The Gibson- Schwartz Model

Gibson and Schwartz in [23] make three assumptions on their commodity price model. The first one is that its price depends only upon the spot price at time t, S_t, where $t \in \mathbb{R}, t \geq 0$, the second, the *instantaneous net convenience yield* of oil, δ, and the third, time to maturity, τ. Also, in [23] they assumes that the spot price of oil and the *net convenience yield* follow a joint diffusion process:

$$dS/S_t = (\mu - \delta) \cdot dt + \sigma_1 \cdot dW_1, \tag{2-1}$$

$$d\delta = \kappa(\mu - \delta) \cdot dt + \sigma_2 \cdot dW_2, \tag{2-2}$$

where dW_1 and dW_2 are correlated increments to standard Brownian processes and $dW_1 \cdot dW_2 = \rho \cdot dt$, where ρ denotes the correlation coefficient between the two Brownian motions and κ is the mean reversion ratio, $\sigma_1, \sigma_2 \in \mathbb{R}$, $\rho \in [-1, 1]$ and $\mu \in \mathbb{R}$.

The *instantaneous net convenience yield* of oil, δ, follows a Ornstein-Uhlenbeck process. If the *instantaneous net convenience yield* of oil, δ, were deterministic and defined as $\delta(S) = \kappa \ln S$, this model would be classified as a one-factor model. If one defines that $X_t = \ln S_t$ and applies Ito's Lemma, Equation (2-1) would be reduced to:

$$dX_t = (\mu - \delta - \sigma_1^2/2) \cdot dt + \sigma_1 \cdot dW_1. \tag{2-3}$$

This model can be written in the risk-neutral version. The main difference between the original model and the risk-neutral one is the parameter λ that is understood as the *convenience yield* risk premium. Also, λ is assumed as a constant.

$$dS/S_t = (\mu - \delta) \cdot dt + \sigma_1 \cdot dW_1^*, \tag{2-4}$$

$$d\delta = (\kappa(\mu - \delta) - \lambda) \cdot dt + \sigma_2 \cdot dW_2^*. \tag{2-5}$$

where dW_1^* and dW_2^* are correlated increments to standard Brownian processes in the risk-neutral version and $dW_1^* \cdot dW_2^* = \rho \cdot dt$.

If one assumes the absence of arbitrage in the market, the future price has to obey to a differential equation:

$$\frac{\sigma_1^2 S^2}{2} \frac{\partial^2 F}{dS^2} + \sigma_1 \sigma_2 \rho S \frac{\partial^2 F}{\partial S \partial \delta} + \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial \delta^2} + (r - \delta) S \frac{\partial F}{\partial S} + (\kappa(\alpha - \delta) - \lambda) \frac{\partial F}{\partial \delta} = \frac{\partial F}{\partial \mathbb{T}}, \tag{2-6}$$

subject to the final condition:

$$F(S, \delta, 0) = 0. \tag{2-7}$$

It can be shown that the solution to this Equation is given by:

$$F(S,\delta,\mathbb{T}) = S\exp\left(-\delta \cdot \frac{(1-e^{-\kappa\mathbb{T}})}{\kappa} + A(\mathbb{T})\right), \tag{2-8}$$

where,

$$A(\mathbb{T}) = \left(r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right)T + \frac{\sigma_2^2(1-e^{-2\kappa\mathbb{T}})}{4\kappa^3} + \left(\hat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right)\frac{(1-e^{-\kappa\mathbb{T}})}{\kappa^2},$$

and

$$\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}.$$

This is one of the most well-accepted models used to evaluate short term financial instruments, such as future contracts, when one might update the estimated value of the market price of risk λ.

In the next section, we will present another popular methodology to forecast spot price of a certain commodity: The Schwartz-Smith Model.

2.6 The Schwartz-Smith Model

Presently, commodity-related securities and projects are evaluated using stochastic models of commodities prices. In [25], a simple two-factor model of commodity prices is presented in order to capture both effects: the short-term deviations and the long-term deviations. It is interesting to say that neither of these two factor is directly observable, but they can be estimated from the spot and future prices using a state space framework and the Kalman filtering.

2.6.1 The Short-Term/Long-Term Model

Let S_t be the spot price of a certain commodity at time t. The spot price will be described by two stochastic variables: $\ln(S_t) = \chi_t + \xi_t$. The equilibrium price level (ξ_t) represents fundamental changes that are expected to persist and the short-term deviations (χ_t) are defined as the difference between spot prices and equilibrium prices and represent temporary changes in prices. Also, the short-term deviations are expected to revert toward zero following a Ornstein-Uhlenbeck process:

$$d\chi_t = -\kappa \cdot \chi_t \cdot dt + \sigma_\chi \cdot dW_\chi, \tag{2-9}$$

where κ is the mean-reversion coefficient that represents the rate at which the short-term deviations are expected to disappear and the time in which a deviation χ_t is expected to halve.

The equilibrium price level (ξ_t) follows a Brownian motion process given by:

$$d\xi_t = -\mu_\xi \cdot dt + \sigma_\xi \cdot dW_\xi, \tag{2-10}$$

where dW_ξ and dW_χ are correlated increments of standard Brownian motion processes with $dW_\xi \cdot dW_\chi = \rho_{\chi\xi} dt$.

In [25], one can say that ξ and χ are jointly normally distributed with mean and covariance matrix:

$$\begin{bmatrix} \chi_t \\ \xi_t \end{bmatrix} \sim N\left(\begin{bmatrix} e^{-\kappa t} \cdot \chi_0 \\ \xi_0 + \mu_\xi \cdot t \end{bmatrix}, \begin{bmatrix} (1 - e^{-2\cdot\kappa t}) \cdot \frac{\sigma_\chi^2}{2\cdot\kappa} & (1 - e^{-\kappa t}) \cdot \frac{\rho_{\xi\chi}\cdot\sigma_\xi\cdot\sigma_\chi}{\kappa} \\ (1 - e^{-\kappa t}) \cdot \frac{\rho_{\xi\chi}\cdot\sigma_\xi\cdot\sigma_\chi}{\kappa} & \sigma_\xi^2 \cdot t \end{bmatrix} \right).$$

2.6.2 Risk-Neutral Processes and Valuation

In [25], the risk-neutral version of the model presented above is rewritten. The risk-neutral valuation is of utmost importance because discounts all cash flows at the risk-free rate and describes the dynamics of the underlying state variables. The risk-neutral stochastic processes are defined as

$$d\chi_t = (-\kappa \cdot \chi_t - \lambda_\chi) \cdot dt + \sigma_\chi \cdot dW_\chi^* \tag{2-11}$$

And the equilibrium level (ξ_t):

$$d\xi_t = (-\mu_\xi - \lambda_\chi) \cdot dt + \sigma_\xi \cdot dW_\xi^*, \tag{2-12}$$

where λ_ξ and λ_χ are new parameters added in the equations in order to represent constant reductions in the drift for each process. Also, dW_ξ^* and dW_χ^* are correlated increments of standard Brownian motion processes with $dW_\xi^* \cdot dW_\chi^* = \rho_{\chi\xi} dt$.

In the risk-neutral version, ξ_t and χ_t are also jointly normally distributed with mean and covariance matrix:

$$\begin{bmatrix} \chi_t \\ \xi_t \end{bmatrix} \sim N\left(\begin{bmatrix} e^{-\kappa t} \cdot \chi_0 - \frac{(1-e^{\kappa t})\cdot\lambda_\chi}{\kappa} \\ \xi_0 + \mu_\xi^* \cdot t \end{bmatrix}, \begin{bmatrix} (1 - e^{-2\cdot\kappa t}) \cdot \frac{\sigma_\chi^2}{2\cdot\kappa} & (1 - e^{-\kappa t}) \cdot \frac{\rho_{\xi\chi}\cdot\sigma_\xi\cdot\sigma_\chi}{\kappa} \\ (1 - e^{-\kappa t}) \cdot \frac{\rho_{\xi\chi}\cdot\sigma_\xi\cdot\sigma_\chi}{\kappa} & \sigma_\xi^2 \cdot t \end{bmatrix} \right),$$

where μ_ξ^* is understood as the drift of the Geometric Brownian motion that the equilibrium prices follow and can be written as $\mu_\xi^* = \mu_\xi - \lambda_\xi$.

In order to value future contracts, let $F_{\mathbb{T}}$ denote the market price for futures contracts with time $\mathbb{T}$ until maturity. Also, it is known that future prices are equal to the expected future spot price under the risk-neutral process and interest rates are supposed deterministic. The future prices can be written as Equation 2-15:

$$\ln(F_{\mathbb{T}}) = \ln(\mathbb{E}^*[S_{\mathbb{T}}]) \tag{2-13}$$

$$= \mathbb{E}^*[\ln(S_{\mathbb{T}})] + \frac{1}{2} \cdot Var^*[\ln(S_{\mathbb{T}})] \tag{2-14}$$

$$= e^{-\kappa t} \cdot \chi_0 + \xi_0 + A(\mathbb{T}), \tag{2-15}$$

where,

$$A(\mathbb{T}) = \mu_\xi^* \cdot \mathbb{T} - (1-e^{-\kappa t}) \cdot \frac{\lambda_\chi}{\kappa} + \frac{1}{2}\left((1-e^{-2\cdot\kappa t}) \cdot \frac{\sigma_\chi^2}{2\cdot\kappa} + \sigma_\xi^2 \cdot \mathbb{T} + 2 \cdot (1-e^{-\kappa t}) \cdot \frac{\rho_{\xi\chi}\cdot\sigma_\xi\cdot\sigma_\chi}{\kappa} \right).$$

The authors show the mathematical equivalence between their model and Gibson-Schwartz using new variables: $\chi_t = \frac{1}{\kappa} \cdot (\delta_t - \alpha)$. In this way, it is possible to transform the Gibson-Schwartz model into the Schwartz-Smith model. One of the differences between these two models is that the Gibson- Schwartz model requires one more parameter (interest rate), but the authors of Schwartz-Smith model discuss and conclude that the interest rate is redundant.

If one replace the risk-free, r, for $r + \Delta$, the convenience yield δ_t, for $\delta_t + \Delta$, the average convenience yield α, for $\alpha + \Delta$ and the spot price drift μ, for $\mu + \Delta$, one will find the same result of X_t and the new δ_t and both follow the same true and risk-neutral processes and lead to the same estimates of the state variables in the short- and long-term model. So, the risk-free rate is not required for specifying the spot price dynamics or valuing future or forward contracts.

3
Risk Measures

In this chapter, we will present some well-known *risk measures* and *performance measures* and its properties. We chose to describe the *Omega Ratio*, the *Value-at-Risk* and the *Conditional Value-at-Risk* due to the fact that those are the most popular and common measures in a risk analysis of a portfolio. These *risk measures* and *performance measures* will be implemented in Chapter 7, where we will optimize the proposed portfolio using these three risk measures and its linear programs.

Investors face the dilemma of how to earn the highest possible future return with minimal risk, by arranging their current wealth over a set of available assets in a portfolio. Markowitz [20] was the pioneer in formulating a mathematical model to solve this allocation problem. He also noticed that a prudent investor aims at maximizing the expected return of an investment, with a lower risk. In the Markowitz model, the risk of a portfolio is measured by the variance of the portfolio return (mean-variance framework).

It is known, in general, that the mean-variance model of Markowitz is not able to capture all of the features of a financial return distribution, given their observed non normality. Instead of mean-variance model, we are going to consider other risk measures for the portfolio solution, such as the *Omega Ratio*, the *Value-at-Risk* and the *Conditional Value-at-Risk*, that will be presented in this chapter.

The choice of a risk measure is an important step towards building a realistic picture of portfolio risk. It is important to conciliate the investor's objectives and the risk in the chosen portfolio.

In the next section we will present the recently proposed performance measure *Omega Ratio*. Also, we will present its equivalent Linear Programming formulation.

3.1 The Omega Ratio

In order to compare the returns of different portfolio strategies, *performance measures*, are used because they capture the downside and upside potential of the constructed portfolio, while remaining consistent with utility maximization.

One of the most well-know *performance measures* is the *Sharpe Ratio*, developed in 1966 by William Sharpe. Due to its simplicity, this measure is very popular and can be pointed out as one of the most referenced risk/return measures used in finance. By definition, for a given set A of assets, the *Sharpe Ratio* is presented as:

$$SR_i = \frac{\mathbb{E}[r_i - r_f]}{\sqrt{V[r_i - r_f]}}, \tag{3-1}$$

where r_i is the return of an asset $i \in A$, r_f is the return of a benchmark (usually risk free rate), $V[r_i - r_f]$ is the variance of the excess return and $\sqrt{V[r_i - r_f]}$ is the standard deviation of the excess return.

The Sharpe Ratio, in general, is used to compare assets or funds. In this way, the higher the Sharpe Ratio, the better the asset/fund is.

As one can see in Equation (3-1), the Sharpe Ratio uses only the first- and second-order moments of the portfolio return and ignores other higher order moments. As a consequence, when the portfolio return is skewed or exhibits fat tails, the Sharpe Ratio may present misleading results in performance evaluations and rankings.

The *Omega ratio* [14] has several important characteristics which can be intuitive and easily understood by the financial market. As an example, this measure is robust to uncertainties in the sample series (unlike the estimators standards), because it is calculated by the observed distribution. Furthermore, it is a proxy of the return's distribution. Basically, it involves the ratio of the returns of the asset gains and losses above or below a threshold and is well known as a measure of financial performance. Usually, the *Omega Ratio* is used to evaluate and compare fixed portfolio strategies.

Let $\tilde{R}$ be the return of a portfolio modeled as a random variable and is assumed to have a cumulative distribution function F_R. The mathematical definition of the *Omega ratio* is described as the probability of having a weighted gain divided by the weighted probability of having a loss on the returns above a threshold τ:

$$\Omega_\tau(\tilde{R}) = \frac{\int_\tau^\infty (1 - F_R(\omega))\, d\omega}{\int_{-\infty}^\tau F_R(\omega)\, d\omega}, \tag{3-2}$$

where τ is a threshold defined by the investor and F_R is the cumulative distribution function of $\tilde{R}$.

The Omega Ratio defines a threshold value τ to distinguish the upside from the downside, i.e, returns above τ are considered profits and returns below τ are considered losses. Usually, the threshold τ is used as the risk-free rate.

The Omega Ratio is a non-convex measure. In [13], it is shown that the

Omega Ratio maximization problem can be reformulated equivalently as a quasi-convex optimization problem. Quasi-convex optimization problems are solved to global optimality in polynomial time. Once reduced to a Linear Program, the portfolio return distribution is approximated by discrete samples. In that way, the Omega Ratio maximization can be used for large-scale portfolios.

In the next subsection, we will deduce the transformation of the Omega Ratio as a non-convex measure to a Linear Program.

3.1.1 Omega Ratio maximization as a Linear Program

Let F_R and f_R describe the cumulative Probability function and the probability density function of the portfolio return random variable $\tilde{R}$. In order to rewrite the Omega Ratio [13] as a Linear Programming maximization we need to develop a more suitable equation for $\Omega_\tau(\tilde{R})$:

$$\Omega_\tau(\tilde{R}) = \frac{\int_\tau^\infty (1 - F_R(\omega))\, d\omega}{\int_{-\infty}^\tau F_R(\omega)\, d\omega}. \tag{3-3}$$

First, we will use integration by parts in Equation (3-3). We will start by the numerator:

$$\int_\tau^\infty (1 - F_R(\omega))\, d\omega = \int_\tau^\infty u(\omega) \cdot dv(\omega),$$

where,

$$u(\omega) = (1 - F_R(\omega))$$
$$v(\omega) = \omega$$

Now, we can rewrite this integral assuming that $\int |\omega| \cdot f_R(\omega) \cdot d\omega < \infty$:

$$\begin{aligned}
\int_{\tau}^{\infty} (1 - F_R(\omega))\, d\omega &= \int_{\tau}^{\infty} u(\omega) \cdot dv(\omega) \\
&= ((1 - F_R(\omega)) \cdot \omega)\big|_{\tau}^{\infty} + \int_{\tau}^{\infty} \omega \cdot f_R(\omega) \cdot d\omega, \\
&= \int_{\tau}^{\infty} \omega \cdot f_R(\omega) \cdot d\omega - (1 - F_R(\tau)) \cdot \tau \\
&= \mathbb{E}_R[\tilde{R}] - \int_{-\infty}^{\tau} \omega \cdot f_R(\omega) \cdot d\omega - (1 - F_R(\tau)) \cdot \tau \\
&= \mathbb{E}_R[\tilde{R}] - \int_{-\infty}^{\tau} \omega \cdot f_R(\omega) \cdot d\omega - \tau \int_{\tau}^{\infty} f_R(\omega) \cdot d\omega \\
&= \mathbb{E}_R[\tilde{R}] - \tau + \mathbb{E}_R[(\tau - \tilde{R})^+] \qquad \text{(3-4)}
\end{aligned}$$

This result was obtained using some algebraic transformations in the Equation (3-4). The integral can be rewritten as:

$$\int_{-\infty}^{\tau} (\tau - \omega) \cdot f_R(\omega) \cdot d\omega = \int_{-\infty}^{\infty} (\tau - \omega) \cdot u_{\leq\tau} \cdot f_R(\omega) \cdot d\omega, \qquad \text{(3-5)}$$

where the function $u_{\leq\tau}$ is a step function, given by:

$$u_{\leq\tau} = \begin{cases} 1 & \text{if } \omega \leq \tau \\ 0 & \text{if } \omega > \tau. \end{cases}$$

So the final result of the integral is:

$$\begin{aligned}
\int_{-\infty}^{\tau} (\tau - \omega) \cdot f_R(\omega) \cdot d\omega &= \int_{-\infty}^{\infty} (\tau - \omega)^+ \cdot f_R(\omega) \cdot d\omega \\
&= \mathbb{E}_R[(\tau - \tilde{R})^+]. \qquad \text{(3-6)}
\end{aligned}$$

Using the same logic of the numerator, the denominator of the Equation (3-3) can be presented in a different way. First, we will integrate by parts the denominator of Equation (3-3):

$$\int_{-\infty}^{\tau} F_R(\omega)\, d\omega = \int_{\tau}^{\infty} u(\omega) \cdot dv(\omega),$$

where,

$$u(\omega) = F_R(\omega)$$
$$\upsilon(\omega) = \omega$$

Now, we can solve this integral:

$$\begin{aligned}
\int_{-\infty}^{\tau} F_R(\omega)\, d\omega &= \int_{-\infty}^{\tau} u(\omega) \cdot d\upsilon(\omega) \\
&= (F_R(\omega) \cdot \omega)\big|_{-\infty}^{\tau} - \int_{-\infty}^{\tau} \omega \cdot f_R(\omega) \cdot d\omega \\
&= F_R(\tau) \cdot \tau - \int_{-\infty}^{\tau} \omega \cdot f_R(\omega) \cdot d\omega \\
&= \int_{-\infty}^{\tau} (\tau - \omega) \cdot f_R(\omega) \cdot d\omega \\
&= \mathbb{E}_R[(\tau - \tilde{R})^+]. \qquad (3\text{-}7)
\end{aligned}$$

The result presented in Equation (3-7) is only obtained using the algebraic transformations presented in Equations (3-5) and (3-6).

Finally, the expression of $\Omega_\tau(\tilde{R})$, Equation (3-3), can be rewritten using the new expressions of the numerator, given by Equation (3-4) and the denominator, Equation (3-7), resulting in :

$$\begin{aligned}
\Omega_\tau(\tilde{R}) &= \frac{\mathbb{E}_R[\tilde{R}] - \tau + \mathbb{E}_R[(\tau - \tilde{R})^+]}{\mathbb{E}_R[(\tau - \tilde{R})^+]}, \\
&= \frac{\mathbb{E}_R[\tilde{R}] - \tau}{\mathbb{E}_R[(\tau - \tilde{R})^+]} + 1. \qquad (3\text{-}8)
\end{aligned}$$

In general, investors aim to find an optimal portfolio allocation that maximizes their wealth respecting their risk profile. Considering that the random portfolio return is defined as:

$$\tilde{R}(\boldsymbol{Q}) = \sum_{i \in A} (\tilde{P}_i^{\,F} - P_i^I) \cdot Q_i, \qquad (3\text{-}9)$$

where

- $\tilde{P}_i^{\,F}$ is the future price of the asset $i \in A$ modeled as a random variable;
- P_i^I is the market price of the asset $i \in A$;

- Q_i is the allocation of asset $i \in A$.

Using Equations (3-8) and (3-9), we can define the portfolio maximization problem as follows:

$$\max_{\boldsymbol{Q} \in \mathcal{Q}} \left\{ \Omega_\tau \left(\sum_{i \in A} (\tilde{P}_i^F - P_i^I) \cdot Q_i \right) \middle| \mathbb{E}_R \left[\sum_{i \in A} (\tilde{P}_i^{\,F} - P_i^I) \cdot Q_i \right] \geq \overline{R}_{min} \right\}, \tag{3-10}$$

where $\boldsymbol{Q} = [Q_1, ..., Q_{|A|}]^T$ is a vector of allocations of each asset $i \in A$, $\mathcal{Q}$ represents the set of feasible decisions and $\overline{R}_{min}$ is the minimum expected return defined by the investor.

Following the classical stochastic programming approach, the continuous random variables $\tilde{P}_i^F$ can be characterized by a set S of possible realizations (scenarios) and its respectively probabilities $\{P_{i,s}^F, p_s\}_{s \in S}$, where p_s is the probability of the scenario s and $P_{i,s}^F$ is the asset realization in the scenario s. Hence, as shown in [3], the mathematical programming problem as specified through Equation (3-10) can be rewritten as:

$$\max_{Q_i} \left[\frac{\sum_{i \in A} \left(\sum_{s \in S} p_s \cdot P_{i,s}^F - P_i^I \right) \cdot Q_i - \tau}{\sum_{s \in S} p_s \left(\tau - \sum_{i \in A} \left(P_{i,s}^F - P_i^I \right) \cdot Q_i \right)^+} \right] \tag{3-11}$$

s.t.

$$Q_i \leq \overline{Q}_i, \qquad \forall i \in A; \tag{3-12}$$

$$Q_i \geq \underline{Q}_i, \qquad \forall i \in A; \tag{3-13}$$

$$\sum_{s \in S} p_s \cdot \left(\sum_{i \in A} (P_{i,s}^F - P_i^I) \cdot Q_i \right) \geq \overline{R}_{min}, \tag{3-14}$$

where the set of feasible decisions is defined as: $\mathcal{Q} = \{\boldsymbol{Q} \in \mathbb{R}^{|A|} \mid \underline{Q}_i \leq Q_i \leq \overline{Q}_i,\ \forall\, i \in A\}$, with $\overline{Q}_i$ and $\underline{Q}_i$, respectively, the maximum and minimum allocation in each asset $i \in A$.

Using an auxiliary variable δ_s to recover the convex function in the objective function denominator, $\left(\tau - \sum_{i \in A} (P_{i,s}^F - P_i^I) \cdot Q_i\right)^+$, the mathematical problem can be restated as

$$\max_{Q_i,\delta_s} \left[\frac{\sum_{i\in A} \left(\sum_{s\in S} p_s \cdot P^F_{i,s} - P^I_i \right) \cdot Q_i - \tau}{\sum_{s\in S} p_s \delta_s} \right] \quad (3\text{-}15)$$

s.t.

$$\delta_s \geq \left(\tau - \sum_{i\in A} (P^F_{i,s} - P^I_i) \cdot Q_i \right), \qquad \forall s \in S \quad (3\text{-}16)$$

$$\delta_s \geq 0, \qquad \forall s \in S \quad (3\text{-}17)$$

$$Q_i \leq \overline{Q}_i, \qquad \forall i \in A; \quad (3\text{-}18)$$

$$Q_i \geq \underline{Q}_i, \qquad \forall i \in A; \quad (3\text{-}19)$$

$$\sum_{s\in S} p_s \cdot \left(\sum_{i\in A} (P^F_{i,s} - P^I_i) \cdot Q_i \right) \geq \overline{R}_{min} \quad (3\text{-}20)$$

Finally, following [4], the mathematical problem presented above is a linear-fractional problem, since it is a maximization of a ratio of affine functions over a polyhedron [13]. Hence, we can state that the Omega Ratio maximization problem as a linear programming as follows:

$$\max_{\theta_i,\eta_s,\kappa} \sum_{i\in A} \sum_{s\in S} \left[(p_s \cdot P^F_{i,s}) - P^I_i \right] \cdot \theta_i - \tau \cdot \kappa \quad (3\text{-}21)$$

s.t.

$$\sum_{i\in A} (P^F_{i,s} - P^I_i) \cdot \theta_i + \eta_s \geq \tau \cdot \kappa, \qquad \forall s \in S \quad (3\text{-}22)$$

$$\eta_s \geq 0 \qquad \forall s \in S \quad (3\text{-}23)$$

$$\theta_i \leq \overline{Q}_i \cdot \kappa \qquad \forall i \in A \quad (3\text{-}24)$$

$$\theta_i \geq \underline{Q}_i \cdot \kappa \qquad \forall i \in A \quad (3\text{-}25)$$

$$\sum_{i\in A} \left[\left(\sum_{s\in S} p_s \cdot P^F_{i,s} \right) - P^I_i \right] \cdot \theta_i \geq \overline{R}_m in \cdot \kappa \quad (3\text{-}26)$$

$$\sum_{s\in S} p_s \cdot \eta_s = 1 \quad (3\text{-}27)$$

$$\kappa \geq 0. \quad (3\text{-}28)$$

Once solved the Linear Programming presented in Equations (3-21), (3-22), (3-23), (3-24), (3-25), (3-26) and (3-27), we obtain the optimal allocation by the ratio of θ_i/κ, i.e., $Q_i = \theta_i/\kappa,\ \forall\ i \in A$.

In the next section, we will present the *Value-at-Risk* and its Linear Programming formulation.

3.2 Value-at-Risk (VaR)

In financial risk management, *Value-at-Risk* (VaR) can be thought of as an index of satisfaction. Its main utility is to measure the risk of loss in a specific portfolio, given a probability and time horizon. Using the same notation of the previous section, we can mathematically define the *Value-at-Risk* as:

$$VaR_\alpha = F_R^{-1}(\alpha) = inf\{r \in \mathbb{R} \mid F_R(r) \geq \alpha\}, \tag{3-29}$$

where α is a fixed level.

The *Value-at-Risk* is the α-quantile of the cumulative probability distribution F_R.

In the subsequent subsection, we will present the VaR maximization as a linear programming.

3.2.1 VaR Maximization as a Linear Programming with integer variables

Although, *Value-at-Risk* plays a important role in risk management in financial markets, this risk measure has some disadvantages, such as being a non-coherent risk measure [1]. In addition, the optimal portfolio problem involving the VaR maximization is a mixed-integer Linear Programming, which, algorithmically, has several complications (such as NP-Hardness).

Once again, we will follow the classical stochastic approach used in the previous section, we define a set S of possible realizations for the portfolio return and its respectively probabilities, p_s. Hence, the mathematical programming of the maximization of the VaR is given by:

$$\max_{\delta, y_s, R_s, Q_i} \delta \tag{3-30}$$

s.t.

$$\delta \leq R_s - y_s \cdot \mathcal{M}, \qquad \forall s \in S \tag{3-31}$$

$$\sum_{s \in S} y_s \leq (1 - \alpha) \cdot |S| - 1 \tag{3-32}$$

$$y_s \in \{0, 1\} \qquad \forall s \in S \tag{3-33}$$

$$\sum_{s \in S} p_s \cdot R_s \geq \overline{R_{min}} \tag{3-34}$$

$$Q_i \leq \overline{Q_i}, \qquad \forall i \in A \tag{3-35}$$

$$Q_i \geq \underline{Q_i}, \qquad \forall i \in A \tag{3-36}$$

$$R_s = \sum_{i \in A} (P^F_{i,s} - P^I_i) \cdot Q_i \qquad \forall s \in S, \tag{3-37}$$

where $R_s, \overline{Q_i}, \underline{Q_i}, P^F_{i,s}, P^I_i$ are the same as described in the previous section. y_s and δ are auxiliary variables created in order to compute the linear programming and $\mathcal{M}$ is a very large number.

The Linear Program described in Equations (3-30), (3-31), (3-32), (3-33), (3-34), (3-35) , (3-36) and (3-37) return an optimal allocation of the maximization of the VaR.

In the following section, we will present the last risk measure that will be used in this dissertation: *Conditional Value-at-Risk* and its linear programing formulation.

3.3 Conditional Value-at-Risk (CVaR)

The *Conditional Value-at-Risk* (CVaR) has been a distinguished risk measure in recent years. The *CVaR* risk metric has been widely used in portfolio problems due to its intuitive structure and its ability to capture the presence of events of high depth (catastrophic) in income distribution.

The *CVaR* can be defined as the conditional expectation of the revenue left-side distribution scenarios, below a given $(1 - \alpha)$ quantile - typically 1-10% (or α from 0.99 to 0.95). The risk level is generally set in order to provide a pessimistic perspective of the results. The mathematical expression is given by:

$$CVaR_\alpha(\tilde{R}) = \mathbb{E}[\tilde{R}|\tilde{R} \leq VaR_\alpha(\tilde{R})] \quad (3\text{-}38)$$
$$= \int_{\{r\in\mathbb{R}|r\leq VaR_\alpha(\tilde{R})\}} \frac{r}{\mathbb{P}\{\tilde{R} \leq VaR_\alpha(\tilde{R})\}} dF_R(r) \quad (3\text{-}39)$$

where $\alpha \in [0,1]$ is the *confidence level*.

By definition, the *CVaR* is the average of the $(1-\alpha)$ worst scenarios of the probability distribution of a given random variable.

With respect to its mathematical properties, *CVaR* is considered a coherent risk measure in the sense of [1]. The proof of the *VaR* and *CVaR* properties are demonstrated in [22].

The subsequent section presents the maximization of the *CVaR*.

3.3.1 CVaR Maximization as a Linear Program

The *CVaR* emerged among other risk measures because of its mathematical and financial properties. Mathematically, it has a number of desirable features, such as being a convex risk measure. Financially, it encourages diversification. Initially, it was calculated as a conditional expected value.

However, after [24], *CVaR* started to be computed as an optimization problem of expected value, subject to linear constraints. This new perspective, enabled its implementation and the popularization of its use in linear optimization problems.

In this sense, [24] proposes an equivalent approach for Equation (3-39) based on the solution of the optimization problem presented in Equation (3-40):

$$\text{CVaR}_\alpha(\tilde{R}) := \sup_z \left\{ z - \frac{\mathbb{E}[(z-\tilde{R})^+]}{(1-\alpha)} \,\middle|\, z \in \mathbb{R} \right\}, \quad (3\text{-}40)$$

where $(x)^+ = \max\{x, 0\}$.

The optimality proof as well as several properties of Equation (3-40) can be found in [24]. The optimization problem can take advantage of some convergence results that are provided for finitely sampled scenarios, such as the convergence of the expectation operator $\left(\lim_{n\to\infty}\left(n^{-1}\sum_{i=1}^n x_i\right) \to \mathbb{E}[\tilde{X}]\right.$, where $\{x_i\}_{i=1}^n$ is a sample series of the random variable $\left.\tilde{X}\right)$ [5].

Therefore, Equation (3-40) can be computed by sampling the exogenous variables. Since the Equation (3-40) is a convex maximization problem. In this sense, for a set S of sampled scenarios with the pair $\{(R_s, p_s)\}_{s\in S}$, the *CVaR* of a continuous random variable $\tilde{R}$ can be approximated by the following linear programming:

$$\text{CVaR}_\alpha(\tilde{R}) \approx \max_{z,\delta_s} \; z - \sum_{s\in S} \frac{p_s \delta_s}{(1-\alpha)} \tag{3-41}$$

subject to:

$$\delta_s \geq z - R_s, \qquad \forall\, s \in S; \tag{3-42}$$

$$\delta_s \geq 0, \qquad \forall\, s \in S; \tag{3-43}$$

$$\sum_{s\in S} p_s \cdot R_s \geq \overline{R_{min}}; \tag{3-44}$$

$$Q_i \leq \overline{Q_i}, \qquad \forall i \in A; \tag{3-45}$$

$$Q_i \geq \underline{Q_i}, \qquad \forall i \in A; \tag{3-46}$$

$$R_s = \sum_{i\in A} (P^F_{i,s} - P^I_i) \cdot Q_i \qquad \forall s \in S. \tag{3-47}$$

The optimal allocation according to the CVaR optimization is obtained by solving the Equations (3-41), (3-42), (3-43), (3-44), (3-45), (3-46) and (3-47).

Finally, the next subsection shows the differences between the most popular risk measures: *VaR* and *CVaR*.

3.3.2 Differences between VaR and CVaR

One criticism made to *VaR* is that it does not differentiate distributions with different expected losses. Generally, both *VaR* and *CVaR* are applied to measure the actual loss of a portfolio. For this reason, they are commonly used for negative financial results. In this context, both are defined as the upper limit for maximum losses allowed. Figure 3.1 illustrates the difference between *VaR* and *CVaR*:

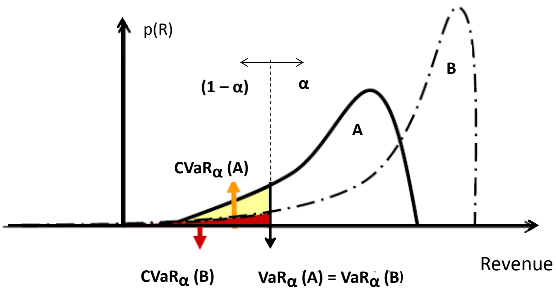

Fig. 3.1: Comparison of CVaR and VaR of two generic probability distributions

Figure 3.1 exemplifies two distributions: A has depth in the lower tail and B with possibilities of events rather negatives. In this example, both distributions show the same ***Value-at-Risk*** of $\alpha\%$, i.e., in both cases with a probability of $\alpha\%$ the result will be higher than the $VaR_\alpha(A)$ and $VaR_\alpha(B)$ value. However, the distribution B has a higher expected value than A. Thus, a model with objective of maximizing the expected return subject to a risk constraint limiting the ***CVaR***, would suggest as optimal solution option A.

4
Principal Component Analysis

In this chapter, we will present the theory of *Principal Component Analysis*. In our work, *Principal Component Analysis* will play an important role by (potentially) decreasing the dimension of the portfolio problem to be defined in Chapter 7. Since all optimization models presented in Chapter 3 depend on a set of simulated scenarios of future prices, it is of utmost importance to develop a method able to generate reliable simulated scenarios. Therefore, the method used in this dissertation is based on the dimensionality reduction of the uncertainties, extracting the correlation structure among the assets in the portfolio.

Principal Component Analysis (PCA) is an important tool for the treatment and analysis of multivariate data. It is a very useful statistical method for studying the covariance structure of a vector of time series, due to the fact that it provides simplification of the original data set, by changing to convenient coordinate frame, potentially decreasing their dimensions and preserving the correlations among random variables.

According to [27], let $\tilde{\boldsymbol{r}} = (\tilde{r_1}, ..., \tilde{r_k})^T$ be a k-dimensional random variable representing the daily log returns of assets i, $i = 1, .., k$ with covariance matrix $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$, given by:

$$\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} = \begin{bmatrix} V[\tilde{r_1}] & Cov(\tilde{r_1}, \tilde{r_2}) & \cdots & Cov(\tilde{r_1}, \tilde{r_k}) \\ Cov(\tilde{r_2}, \tilde{r_1}) & V[\tilde{r_2}] & \cdots & Cov(\tilde{r_2}, \tilde{r_k}) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(\tilde{r_k}, \tilde{r_1}) & Cov(\tilde{r_k}, \tilde{r_2}) & \cdots & V[\tilde{r_k}] \end{bmatrix},$$

where $Cov(\tilde{r_i}, \tilde{r_j}) = \mathbb{E}[(\tilde{r_i} - \mu_i)(\tilde{r_j} - \mu_j)]$, $V[\tilde{r_i}] = \mathbb{E}[(\tilde{r_i} - \mu_i)^2]$ and $\mu_i = \mathbb{E}[\tilde{r_i}]$, $i = 1, .., k$.

Principal Component Analysis (PCA) aims at using as little as possible linear combinations of $\{\tilde{r_i}\}_{i=1}^k$ to explain the structure of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$. If $\{\tilde{r_i}\}_{i=1}^k$ denotes the daily log returns of k assets, then the PCA is recommended to be applied to this series in order to study the sources of changes on these k asset returns.

PCA can either be applied in the covariance matrix $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$ or to the correlation matrix $\boldsymbol{\rho}_{\tilde{\boldsymbol{r}}}$ of $\tilde{\boldsymbol{r}}$. The latter can be defined as $\boldsymbol{\rho}_{\tilde{\boldsymbol{r}}} = \boldsymbol{D}^{-1} \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{D}^{-1}$. The relationship between $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$ and $\boldsymbol{\rho}_{\tilde{\boldsymbol{r}}}$ is that the correlation matrix is the covariance matrix of the standardized random vector $\tilde{\boldsymbol{r}}^* = \boldsymbol{D}^{-1} \cdot \tilde{\boldsymbol{r}}$, where $\boldsymbol{D}$ is the diagonal matrix of standard deviations of the components of $\tilde{\boldsymbol{r}}$, that is $\boldsymbol{D} = diag\{S_1, S_2, .., S_k\}$, where $S_i = \sqrt{\mathbb{E}[(\tilde{r_i} - \mu_i)^2]}$, $i = 1, .., k$.

Let $\boldsymbol{e_i} = (e_{i1}, ..., e_{ik})^T \in \mathbb{R}^k$ for $i = 1, ..., k$, and $\tilde{\boldsymbol{r}} \in \mathbb{R}^k$ be a k-dimensional random variable representing the daily log returns of assets i, $i = 1, .., k$. Then we can form the linear combination:

$$\tilde{\beta}_i = \boldsymbol{e}_i^T \cdot \tilde{\boldsymbol{r}} = \sum_{j=1}^{k} e_{ij} \cdot \tilde{r}_j, \tag{4-1}$$

where $\tilde{\beta}_i$ is the i-th principal component also known as *scores* and $\boldsymbol{e}_i^T$ are the coordinates of the linear combination also known as the *loadings*.

Hence, the main idea of PCA is to preserve as much as possible of the variation of the original random variables $\tilde{\boldsymbol{r}}$ in a lower dimension random vector, so that we simplify our problem by reducing its dimension. In this sense, following [12], PCA can be accomplished by maximizing the variance of the principal components sequentially, i.e., $\max_{\boldsymbol{e}_1} \{V[\tilde{\beta}_1]\}$, then $\max_{\boldsymbol{e}_2} \{V[\tilde{\beta}_2]\}$, $\ldots$, then $\max_{\boldsymbol{e}_k} \{V[\tilde{\beta}_k]\}$, keeping them uncorrelated, i.e. $Cov(\tilde{\beta}_i, \tilde{\beta}_j) = 0, \; \forall \, i, j = 1, \ldots k, \; i \neq j$. In practice, only a few principal components will have relevant variance, thus we can work only with these components.

The main difficulty in this approach is to find the vector $\boldsymbol{e}_i^T$ that maximizes the variance of the i-th principal component. Next we show a procedure that optimally find these vectors, for each $i = 1, \ldots, k$. Starting with the first component, the mathematical problem can be stated as:

$$\max_{\boldsymbol{e}_1}\{V[\tilde{\beta}_1]\} = \max_{\boldsymbol{e}_1}\{V[\boldsymbol{e}_1^T \cdot \tilde{r}]\} = \max_{\boldsymbol{e}_1}\{\boldsymbol{e}_1^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1\}. \tag{4-2}$$

Performing the maximization problem (4-2), the optimal solution will not be achieved for finite $\boldsymbol{e}_1$. Therefore, it is common to introduced some normalization constraint, such that $\boldsymbol{e}_1^T \boldsymbol{e}_1 = 1$ or $\max_j \{|e_{1,j}|\} = 1$. The choice of the constraint is dependent on the circumstances. For our purposes, we choose to use $\boldsymbol{e}_1^T \boldsymbol{e}_1 = 1$ for a specific reason discussed later. So, the maximization problem (4-2) becomes:

$$\max_{\boldsymbol{e_1}}\{\boldsymbol{e}_1^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1 \mid \boldsymbol{e}_1^T \boldsymbol{e}_1 = 1\}. \tag{4-3}$$

In order to solve (4-3), we can use the first-order condition for the Lagrangian function, i.e. maximize $\{\boldsymbol{e}_1^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1 - \lambda \cdot (\boldsymbol{e}_1^T \boldsymbol{e}_1 - 1)\}$, where λ is the Lagrange multiplier. Differentiating with respect to $\boldsymbol{e}_1$ gives

$$(\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} - \lambda \cdot \mathbb{I}_k) \cdot \boldsymbol{e}_1 = \boldsymbol{0} \Rightarrow \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1 = \lambda \cdot \boldsymbol{e}_1, \tag{4-4}$$

where $\mathbb{I}_k$ is a $k \times k$ identity matrix. Thus, λ is an eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$ and $\boldsymbol{e}_1$ its corresponding eigenvector.

To define which eigenvalue (eigenvector) should be chosen, note that the original objective function in Equation (4-3) is $\boldsymbol{e}_1^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1$. Therefore, we have that

$$\boldsymbol{e}_1^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1 = \boldsymbol{e}_1^T \lambda \boldsymbol{e}_1 = \lambda \cdot (\boldsymbol{e}_1^T \boldsymbol{e}_1) = \lambda \tag{4-5}$$

Thus, λ must be as large as possible to maximize (4-3). So, we choose the largest eigenvalue and $\boldsymbol{e}_1$ is the eigenvector associated with the largest eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$. It is important to justify why we choose $\boldsymbol{e}_1^T \boldsymbol{e}_1 = 1$ as the normalization constraint. The reason is described in (4-5). The conclusion that the optimal Lagrange multiplier is equal to the largest eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$ is only valid if this constraint is chosen. Otherwise, the result is not valid and another procedure must be done. Since evaluating the eigenvalues of a semi-definite matrix is not computationally burden, it seems to be a good choice to use $\boldsymbol{e}_1^T \boldsymbol{e}_1 = 1$.

After finding the coordinates of the first principal component $(\boldsymbol{e}_1)$, we can implement the same procedure to find the coordinates for the remaining principal components. However, we need to guarantee that the principal components are uncorrelated, i.e. $Cov(\tilde{\beta}_i, \tilde{\beta}_j) = 0,\ \forall\, i, j = 1, \ldots, k,\ i \neq j$. In this sense, for the second component, the mathematical problem can be stated as:

$$\max_{\boldsymbol{e_2}} \{\boldsymbol{e}_2^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_2 \mid \boldsymbol{e}_2^T \boldsymbol{e}_2 = 1,\ Cov(\tilde{\beta}_1, \tilde{\beta}_2) = 0\} \tag{4-6}$$

However, note that

$$Cov(\tilde{\beta}_1, \tilde{\beta}_2) = \boldsymbol{e}_2^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1 = \boldsymbol{e}_2^T \cdot \lambda_1 \cdot \boldsymbol{e}_1 = \lambda_1 \boldsymbol{e}_2^T \cdot \boldsymbol{e}_1 = 0 \tag{4-7}$$

Assuming that $\lambda_1 > 0$, since it is the largest eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$, we can substitute the constraint $Cov(\tilde{\beta}_1, \tilde{\beta}_2) = 0$ in (4-6) by $\boldsymbol{e}_2^T \cdot \boldsymbol{e}_1 = 0$. Using, again, the first-order condition of the Lagrange function, we can maximize $\{\boldsymbol{e}_2^T \cdot \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_1 - \lambda \cdot (\boldsymbol{e}_2^T \boldsymbol{e}_2 - 1) - \varphi \cdot (\boldsymbol{e}_2^T \cdot \boldsymbol{e}_1)\}$, where λ and φ are the Lagrange multipliers of (4-6).

Differentiating with respect to $\boldsymbol{e}_2$ gives

$$\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_2 - \lambda \cdot \boldsymbol{e}_2 - \varphi \cdot \boldsymbol{e}_1 = \mathbf{0} \Rightarrow \boldsymbol{e}_1^T \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_2 - \lambda \cdot \boldsymbol{e}_1^T \cdot \boldsymbol{e}_2 - \varphi \cdot \boldsymbol{e}_1^T \cdot \boldsymbol{e}_1 = \mathbf{0} \quad (4\text{-}8)$$

Since $\boldsymbol{e}_1^T \boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_2 = 0$, $\boldsymbol{e}_1^T \cdot \boldsymbol{e}_2 = 0$ and $\boldsymbol{e}_1^T \cdot \boldsymbol{e}_1 = 1$, we have that $\varphi = 0$. Hence,

$$\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} \cdot \boldsymbol{e}_2 - \lambda \cdot \boldsymbol{e}_2 = \mathbf{0} \Rightarrow (\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}} - \lambda \cdot \mathbb{I}_k) \cdot \boldsymbol{e}_2 = \mathbf{0} \quad (4\text{-}9)$$

Again, λ is an eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$ and $\boldsymbol{e}_2$ is its correspondent eigenvector. Performing the same analysis of the first component, we have that $\boldsymbol{e}_2$ is the eigenvector associated with the second largest eigenvalue of $\sigma_{\tilde{r}}$.

Repeating with this procedure, we can find all the coordinates of the linear combination that defines each principal component, concluding that each $\boldsymbol{e}_i$ is the eigenvector associate with the i-th largest eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$ (we refer to [12] for the complete proof and a wider discussion of this procedure). In order to complete this transformation, the principal components relates to the original random vector by the following equation:

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{E}^T \cdot \tilde{\boldsymbol{r}}. \quad (4\text{-}10)$$

where $\boldsymbol{E}^T = [e_1, e_2, \ldots, e_k]^T$ is the matrix with the eigenvectors on its columns and ordered by the highest to the smaller eigenvalue, (also known as *matrix of loadings*).

In addition, we have that

$$\begin{aligned} \sum_{\boldsymbol{i=1}}^{\boldsymbol{k}} V[\tilde{r}_i] &= tr(\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}) \\ &= \sum_{\boldsymbol{i=1}}^{\boldsymbol{k}} \lambda_i \\ &= \sum_{\boldsymbol{i=1}}^{\boldsymbol{k}} V[\tilde{\beta}_i]. \end{aligned} \quad (4\text{-}11)$$

Thus, rewriting (4-11):

$$\frac{V[\tilde{\beta}_i]}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{k}} V[\tilde{r}_i]} = \frac{\lambda_i}{\lambda_1 + ... + \lambda_k} \qquad i = 1, 2, ..., k. \quad (4\text{-}12)$$

Consequently, the proportion of total variance in $\tilde{\boldsymbol{r}}$ explained by the *i*-th principal component is simply the ratio between the *i*-th eigenvalue and the sum of all eigenvalues of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$. One can also compute the cumulative proportion of total variance explained by the first i principal components, i.e., $\sum_{j=1}^{i} \lambda_j / \sum_{j=1}^{k} \lambda_j$.

In practice, one selects a small i such that the prior cumulative proportion is large. Since $tr(\boldsymbol{\rho}_{\tilde{\boldsymbol{r}}}) = k$, the proportion of variance explained by the *i*-th principal component becomes λ_i/k when the correlation matrix is used to perform the PCA.

A by-product of PCA is that a zero eigenvalue of $\boldsymbol{\sigma}_{\tilde{\boldsymbol{r}}}$, or $\boldsymbol{\rho}_{\tilde{\boldsymbol{r}}}$, indicates the existence of an *exact* linear relationship between the components of $\tilde{\boldsymbol{r}}$. For instance, if the smallest eigenvalue is $\lambda_k = 0$, then $V[\tilde{\beta}_k] = 0$. Therefore, $\tilde{\beta}_k = \sum_{\boldsymbol{i=1}}^{\boldsymbol{k}} e_{kj}\tilde{r}_j$ is a constant random variable and there are only $k-1$ random quantities in $\tilde{\boldsymbol{r}}$. In this case, the dimension of $\tilde{\boldsymbol{r}}$ can be reduced. For this reason, PCA has been used in the literature as a tool for dimension reduction. One only must define an acceptance level. Thus, all principal components which the variance ratio (4-12) is lower that this level is assumed to be constant and then disregarded of the analysis, reducing the dimension of the problem.

5
Backtesting Techniques

This chapter outlines the theory of the *Backtesting* and its importance in the validation of a portfolio. In Chapter 7, we will implement the backtesting techniques in our portfolio in order to validate our proposed method.

Risk models are used as an important instrument in decision making by investment managers that intend to adjust the relationship between the portfolio return and the risk incurred. It is also relevant for authorities regulators, who must observe whether financial institutions are taking more risks than its assets can support. Therefore, loss estimates provided by risk models must be constantly evaluated through *backtesting*, which compares the estimated risk with losses incurred in fact.

The most popular methods for *backtesting*, as proposed in [6], [7] and [18], analyze the number of violations of the *VaR*. A violation takes places when a return value surpasses the lower limit of the *VaR* estimated by the risk model. As the probability of violation of the *VaR* level α equals α, then one can say that the risk model is adequate. The number of violations is modeled by a sequence of i.i.d. random variables of Bernoulli type with parameter α. The works cited above use likelihood ratio tests to test the adopted coverage, $1 - \alpha$.

By disregarding the magnitude of losses, the above mentioned *backtesting* methods based on violations series cannot be applied to the Expected Shortfall. There are few studies in the literature *backtesting* on this risk measure. Kerkhof and Melenberg developed a framework for *backtesting* any risk measure [15], which consists on a hypothesis test. The statistic of this test is the difference between the measure of the risk from the model and the measure applied to the historical distribution of losses.

One of the problems of *backtesting* methods that will be discussed is the low rate of rejection of the models that are poorly specified. As we will see below, the power of the tests can be undesirably low when the sample size used in the backtest is small.

In order to simplify the presentation of the methods, we will take the time horizon to be one day.

5.1 Backtest based on violations

Let $\{R_t(\boldsymbol{Q})\}_{t=1}^{\mathbb{T}}$ be the series of observed portfolio returns for a given allocation $(\boldsymbol{Q})$ and $\{VaR_t\}_{t=1}^{\mathbb{T}}$ be the series of estimatives of *VaR* with level α. A violation happens when the portfolio loss in a time *t* is larger than the loss estimated by the *VaR* at the same time *t*. So, the sequence of violations $\{R_t(\boldsymbol{Q})\}_{t=1}^{\mathbb{T}}$ is given by:

$$I_t = \begin{cases} 1, & \text{if } R_t(\boldsymbol{Q}) < -VaR_t \\ 0, & \text{otherwise} \end{cases} \tag{5-1}$$

If the *VaR* model was correctly specified, the probability of loss $R_t(\boldsymbol{Q})$ be larger than $\{VaR\}_t$ is equal to α. Thus, it is expected that the time series of I_t be i.i.d. with Bernoulli distribution with parameter α.

The Kupiec test is classified as a test of unconditional coverage, and the tests proposed by Christoffersen, [7] and [18], are independence and conditional coverage tests.

5.1.1 Kupiec's Test

Kupiec's test [18] consists of a hypotesis test whether the observed frequency of exceptions is consistent with the frequency of expected exceptions according to the VaR model and a chosen confidence interval, on coverage, $1 - \alpha$.

$$\begin{cases} H_0 & : \theta = \alpha \\ H_1 & : \theta \neq \alpha \end{cases}. \tag{5-2}$$

Under the null hypothesis that the model is correct, I_t follows a Bernoulli distribution with parameter α. Therefore, the total violations V has a binomial distribution:

$$V = \sum_{t=1}^{\mathbb{T}} I_t \sim Binomial(\mathbb{T}, \alpha) \tag{5-3}$$

The author proposes the use of the likehood ratio test in order to test the null hypothesis. The statistics is given by:

$$\Lambda(V) = -2 \cdot \ln\left(\frac{\mathcal{L}(\alpha|V)}{\sup_\theta\{\mathcal{L}(\theta|V) : \theta \in [0,1]\}}\right)$$

$$= \begin{cases} -2 \cdot \ln\left(\dfrac{\alpha^V \cdot (1-\alpha)^{(\mathbb{T}-V)}}{\hat{\alpha}^V \cdot (1-\hat{\alpha})^{(\mathbb{T}-V)}}\right) & , if \ \ V > 0, \\ -2 \cdot \ln((1-\alpha)^{\mathbb{T}}) & , if \ \ V = 0, \end{cases}$$

where $\mathcal{L}$ is the likelihood function, $\hat{\alpha} = V/\mathbb{T}$ is the maximum likelihood estimator of α and $\Lambda(V) \sim \chi^2(1)$.

In a likelihood ratio test, the test statistic is asymptotically distributed as the chi-squared. The number of degree of freedom is given by the difference between the number of free parameters in the model associated with the null and alternative hypothesis. The model of the null hypothesis has no free parameters, because it is assumed that $\theta = \alpha$. In the alternative hypothesis, the parameter θ is free.

5.2 Christoffersen's Test

According to [19], large changes tend to be followed by large changes -of either sign- and small changes tend to be followed by small changes. Thus, it is desirable that the *VaR* model would be able to capture this stylized fact. If the model considers the volatility as a constant it is likely that the violations would occur with a higher frequency than expected in periods of higher volatility and with lower frequency in the other periods of time.

Christoffersen proposes in [6] and [7] a statistical test in order to verify if the violation series is temporarily independent. The test indicates whether the risk model was able to capture the variation of the volatility of the series.

In [6], the author makes a simplification of the given problem and tests the independence between two dates in a row in the series of violations. If this independence exists, and if the non-consecutive dates are independent as well, the series can be understood as a first-order Markov Chain. The matrix of transition probabilities is given by:

$$\Pi = \begin{bmatrix} (1-\alpha_{01}) & (1-\alpha_{11}) \\ (\alpha_{01}) & (\alpha_{11}) \end{bmatrix}, \tag{5-4}$$

where $\alpha_{ij} = P(I_t = j|I_{t-1} = i)$. This means that α_{11} is the probability of happening a violation given that one has already happened and α_{01} is the probability of a violation given that no violation has occurred in the day before $t - 1$.

The independence test aims verifying if the probability of having a violation at the date t does not depend of having had a violation at $t - 1$. The hypothesis can be written as shown in:

$$\begin{cases} H_0 & : \alpha_{01} = \alpha_{11}, \\ H_1 & : \alpha_{01} \neq \alpha_{11}. \end{cases} \tag{5-5}$$

Alternatively, it is possible to perform a conditional coverage test using a similar hypothesis test:

$$\begin{cases} H_0 & : \alpha_{01} = \alpha_{11} = \alpha, \\ H_1 & : \alpha_{01} \neq \alpha_{11}. \end{cases} \tag{5-6}$$

So, let:

$$T_0 = \sum_{t=1}^{T-1} (1 - I_t), \qquad T_{01} = \sum_{t=1}^{T-1} I_{t+1} \cdot (1 - I_t),$$
$$T_1 = \sum_{t=1}^{T-1} I_t, \qquad T_{11} = \sum_{t=1}^{T-1} I_{t+1} \cdot I_t,$$

where:

- T_0 is the number of non-violations and T_1 is the number of violations, both of them disregarding the last element of the series I_t;
- T_{01} is the number of violations that happened after a non-violation;
- T_{11} is the number of violations that happened after a violation;

It is interesting to say that T_{01} and T_{11} have a binomial distribution.

In order to test the independence hypothesis in Equation (5-5), the likelihood ratio test can be presented as:

$$\Lambda_{IND}(V) = -2 \cdot ln\left(\frac{\sup_{(\alpha_{01},\alpha_{11})} L(\alpha_{01},\alpha_{11}|I) : \alpha_{01} = \alpha_{11}}{\sup_{(\alpha_{01},\alpha_{11})} L(\alpha_{01},\alpha_{11}|I)}\right), \tag{5-7}$$

where I represents the observation of T_0, T_1, T_{01} and T_{11}. $\Lambda(V)$ has asymptotic distribution $\chi^2(1)$. In this way, we have that:

- $\hat{\alpha} = (T_0 + T_{11})/(T_0 + T_1)$ is the maximum likelihood estimator of $\hat{\alpha}$ in the null hypothesis.
- $\hat{\alpha_{01}} = T_{01}/T_0$ and $\hat{\alpha_{11}} = T_{11}/T_1$ are the maximum likelihood estimators of $\hat{\alpha_{01}}$ and $\hat{\alpha_{11}}$, respectively.

The test statics is given by:

$$\Lambda_{IND}(V) = \begin{cases} -2 \cdot \ln\left(\frac{\hat{\alpha}^{(T_{01}+T_{11})} \cdot (1-\hat{\alpha})^{(T_0+T_1-T_{01}-T_{11})}}{\hat{\alpha_{01}}^{T_{01}} \cdot (1-\hat{\alpha_{01}})^{(T_0-T_{01})} \cdot (1-\hat{\alpha_{11}})^{(T_1-T_{11})}}\right) & ,if\ T_{01}+T_{11} > 0 \\ -2 \cdot \ln\left(\frac{\hat{\alpha}^{T_{01}} \cdot (1-\hat{\alpha})^{(T_0+T_1-T_{01})}}{\hat{\alpha_{01}}^{T_{01}} \cdot (1-\hat{\alpha_{01}})^{(T_0-T_{01})}}\right) & ,if\ T_{11} = 0 \\ -2 \cdot \ln\left(\frac{\hat{\alpha}^{T_{11}} \cdot (1-\hat{\alpha})^{(T_0+T_1-T_{11})}}{\hat{\alpha_{11}}^{T_1} \cdot (1-\hat{\alpha_{11}})^{(T_1-T_{11})}}\right) & ,if\ T_{01} = 0 \end{cases}$$

This test is only valid when there is at least one violation in the observed series.

If one wishes to test the null hypothesis $\alpha_{01} = \alpha_{11} = \alpha$, one should replace $L(\hat{\alpha}|I)$ for $L(\alpha|I)$. The statistics has distribution $\chi^2(2)$.

6
Method for constructing and optimizing commodities portfolio

In this chapter, we will outline the method used in this work. We will explain the criteria used to calculate the daily log returns and their distribution, the dimension reduction and the reconstruction of the prices of the equities used in the case studies. The data was obtained from sources such as *Bloomberg, Platts and Reuters* from September 14th, 2011 until December 27th, 2013. The data manipulation was done using the free software R.

In order to illustrate our methodology, we present a diagram of the methodology in Figure 6.1:

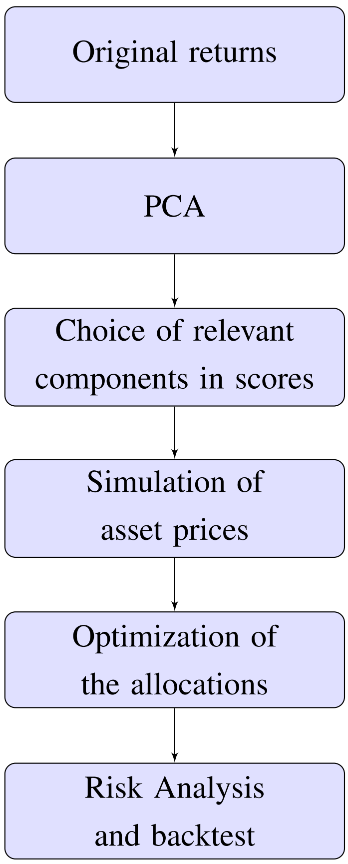

Fig. 6.1: Diagram of the method

Figure 6.1 presents the method that will be implement in Chapter 7. In possession of daily log returns of each asset r_i, $i \in A$, we will apply the PCA in these returns in order to decrease its dimensionality. Once chosen the most relevant principal components, we will model individually each principal component chosen and we will simulate the returns and, consequently the prices. Subsequently, we will optimize the allocations of the portfolio using the risk measures presented in Chapter 3 and its linear programs. Finally, we will test the performance of the optimal allocations.

In the next sections, each of the steps illustrated in Figure 6.1 will be explained.

6.1 Original Returns

Let us assume P_t to be the price of an asset at a generic time t, and $P_{t-\tau}$, the price of an asset at an earlier time $t-\tau$. The *log return* at time t for a horizon time τ is defined as:

$$r_{t,\tau} \equiv ln\left(\frac{P_t}{P_{t-\tau}}\right), \tag{6-1}$$

where, in general, $\tau = 1$, because we are using daily prices of commodities.

Following [21], we remark that:

1. The distribution of the log returns can be projected to any horizon and then transformed back into the distribution of market prices at the specified horizon.

2. Log returns have an approximately symmetrical distribution, unlike the distribution of linear returns or total returns. This is an advantage when it comes to model the distribution of the log returns.

In the next section, we will present PCA applied to the portfolio problem and the choice of the most relevant components in the scores.

6.2 PCA and Most Relevant Components

When a portfolio has a long number of assets, it is wise to provide dimensionality reduction before modeling the return series and optimizing the portfolio. This can be done by applying *Principal Component Analysis* in the returns of each asset of the portfolio, as presented in Chapter 4. This will be performed in our Case Study of Chapter 7. We applied *Principal Component Analysis* in order to (potentially) reduce the dimension of our problem.

Let $\boldsymbol{r} = (r_1, \ldots, r_k)$ be the observed daily log returns of k assets and p periods of time, so $\boldsymbol{r}$ is $p \times k$ matrix. The *Principal Components Analysis* decomposition of $\boldsymbol{r}$ can therefore be given as

$$\boldsymbol{\beta} = \boldsymbol{r} \cdot \boldsymbol{E}, \tag{6-2}$$

where $\boldsymbol{E}$ is the so-called matrix of *loadings* and a $k \times k$ matrix whose columns are the eigenvectors of the sample covariance matrix of $\boldsymbol{r}$, $\boldsymbol{\beta}$ is a $p \times k$ matrix, also known as *scores* as presented in Chapter 4.

The results of PCA can be summarized in two elements: *scores* and *loadings*. As explained in Chapter 4, the loadings are the weights, the scores are the original series translated to another coordinate system.

As discussed in Chapter 4, by defining a confidence level, we will apply the Equation (4-12) in order to decide the most relevant components in the scores, those whose variance are relevant. Most of times, few principal components can explain almost the whole time series.

Therefore, only the most relevant components need to be kept in order to decrease the dimension of the portfolio problem. Let n be the number of chosen components, where $n \leq k$. This truncated transformation can be given as

$$\boldsymbol{\beta}_i = \boldsymbol{r} \cdot \boldsymbol{E}_i, \ \forall i = 1, ..., n \tag{6-3}$$

where the truncated scores, given by $\boldsymbol{\beta}_i$ is a $p \times n$ matrix, with $n \leq k$. Also, the loading matrix, $\boldsymbol{E}_i$, will be a $k \times n$ matrix, i.e. the n first columns of $\boldsymbol{E}$.

This procedure aims to potentially reduce the dimension of the portfolio and yet is able to keep the correlation structure among the assets.

Since the remaining scores, $\{\boldsymbol{\beta}_i\}_{i=1}^{n}$, keep almost all the dynamics of the original returns $\boldsymbol{r}$, we can use these variables to perform our analysis taking advantage of the resulting dimensionality reduction obtained by applying PCA to the original returns of each asset in the portfolio.

In the next couple of sections, we will present a method to simulate the log-returns and, consequently, the prices of the assets, using the reduced score matrix, $\{\boldsymbol{\beta}_i\}_{i=1}^{n}$.

6.3 Simulation of asset prices

As discussed in Chapter 3, aiming to find the optimal portfolio using the optimization models presented, we need to obtain a set of simulated scenarios that represent the uncertainties (e.g. future asset prices) in the investor's portfolio.

In this section, we present a method based on the chosen scores, $\{\boldsymbol{\beta}_i\}_{i=1}^n$, and the most commons processes used to model scores devised from return series, namely: *GARCH* with *Normal* or *Student t* innovations [11], *Geometric Brownian Motion* [16] and *Normal distribution*. Since the scores, by definition, are uncorrelated, we can calibrate/simulate them using univariate models. Then, after simulating the chosen scores we can obtain the log returns using the transpose of the reduced loading matrix, defined in Equation (6-3), and finally obtain the simulated prices.

We decide to model chosen scores, $\{\boldsymbol{\beta}_i\}_{i=1}^n$ using GARCH models and Geometric Brownian Motion due to their properties. Those definitions will be presented subsequently.

Since time-varying volatility is more common than constant volatility in the series, generalized autoregressive conditional heteroskedasticity (GARCH) models emerged as a solution. The GARCH model has the ability to model conditional variances $\sigma_{t,i}$, i.e., assumes that the changes in variance are a product of the realizations of preceding errors.

The uni-variate GARCH model is presented as follows: For a score series $\beta_{t,i}, \quad \forall t = 1, ..., p, i = 1, ..., n$, where n is the number of chosen scores, let $a_{t,i} = \beta_{t,i} - \mu_{t,i}$, be the innovation at time t for the chosen score $\beta_{t,i}$ is:

$$a_{t,i} = \sigma_{t,i} \cdot \epsilon_{t,i}, \qquad \sigma_{t,i}^2 = \theta_{i,0} + \sum_{k=1}^{m} \theta_{i,k} \cdot a_{t-k,i}^2 + \sum_{j=1}^{q} \gamma_{i,j} \cdot \sigma_{t-j,i}^2, \tag{6-4}$$

where $\mu_{t,i}$ is the conditional mean of each asset $i \in A$, $\{\epsilon_{t,i}\}$ is a sequence of i.i.d. random variables with mean 0 and variance 1, $\{\theta_{i,k}\}_{k=0}^m > 0$, $\{\gamma_{i,j}\}_{j=1}^s \geq 0$ and $\sum_{l=1}^{max(m,q)} (\theta_{i,l} + \gamma_{i,l} < 1)$, $i = 1, ..., n$. Usually, $\{\epsilon_{t,i}\}$ is assumed to follow a standard normal or standardized Student t(m).

One of the main features of GARCH model is that the shock $a_{t,i}$ of the score $\boldsymbol{\beta}_i, i = 1, ..., n$ is serially uncorrelated, but dependent and its dependence can be described by a quadratic function of its lagged values.

The estimation of the parameters of the *GARCH (m,q)* model was done by using the software R. After estimation, it is important to test the *residuals* and the *Squared Residuals* of the *GARCH (m,q)* chosen, in order to validate the model. The statistical tests used were: *Auto-correlation Function* (ACF) and *Partial Correlation Function* (PACF).

We will also use, the *Geometric Brownian Motion* process, as an alternative way to model the scores and, as consequence, simulate the prices. This model is usually used in mathematical finance and can be defined as a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift. Thus, for a score $\boldsymbol{\beta}_i, i = 1, ..., n$, its expression is given by:

$$\Delta\beta_{t,i} = \mu_i \cdot \Delta t + \sigma_i \cdot \epsilon_{t,i} \cdot \sqrt{\Delta t}, \tag{6-5}$$

where Δt is the time period, μ_i is the mean of $\boldsymbol{\beta}_i$, σ_i is the standard deviation of the score $\boldsymbol{\beta}_i$ and $\epsilon_{t,i}$ is a random normal with mean 0 and variance 1.

Finally, we are able to simulate τ steps ahead the scores $\{\boldsymbol{\beta}_i\}_{i=1}^n$ by sampling the innovations $\epsilon_{t,i}$ and applying the estimated model. With a set of simulated scenarios S, we can use the transpose of the matrix of loadings to obtain the simulation of the log returns τ steps forward:

$$r_{t+\tau,s} = \boldsymbol{\beta}_{t+\tau,s} \cdot \boldsymbol{E}^T, \qquad \forall \tau \in \mathbb{T}, s \in S \tag{6-6}$$

where $\mathbb{T}$ is the set of step forward periods simulated, $r_{t+\tau,s}$ is the matrix of simulated log returns τ steps forward, $\boldsymbol{E}^T$ is the transpose of the matrix of loadings and $\boldsymbol{\beta}_{\boldsymbol{r}_{t+\tau,s}}$ is the matrix of scores τ steps forward. Note that, both $\boldsymbol{\beta}_{\boldsymbol{r}_{t+\tau,s}}$ and $\boldsymbol{E}^T$ have a reduced dimension.

Finally, for a given asset $i \in A$, τ steps forward the estimated prices will be simulated as follows:

$$r_{t+\tau,i,s} \equiv ln\left(\frac{P_{i,t+\tau,s}}{P_{i,t+\tau-1,s}}\right) \Rightarrow P_{i,t+\tau,s} = \exp(r_{t+\tau,i,s}) \cdot P_{i,t+\tau-1,s} \quad \forall \tau \in \mathbb{T}, s \in S, \tag{6-7}$$

where:

- $P_{i,t,s} = P_{i,t}$, where $P_{i,t}$ is the last observed price for the asset i in the historical data;

- $r_{t+\tau,i,s}$ is the simulated return of the asset $i \in A$, in the step $\tau \in \mathbb{T}$;
- $P_{i,t+\tau,s}$ is the simulated price of the asset $i \in A$, in the step $\tau \in \mathbb{T}$;
- $\mathbb{T}$ and S have the same interpretation as in Equation (6-6).

This set of simulated prices will be used in the optimization models described in Chapter 3 in order to find the optimal asset allocation for each risk measure. In the next section, we discuss how the optimization is performed.

6.4 Optimization of the allocations

The allocations used in the construction of the portfolio, in $US\$$, were decided by maximizing the ***CVaR***, ***VaR*** and ***Omega Ratio***. Each optimization model has its own Case Study in Chapter 7.

In possession of the simulated prices (described in the last section), we optimize the allocations using CVaR, VaR and Omega Ratio, as described in Chapter 3. The following Equations illustrate how the portfolio revenue is calculated and then, how the Linear Programming is formulated:

$$R_s = \sum_{i \in A} (P^F_{i,|\mathbb{T}|,s} - P^I_i) \cdot Q_i, \tag{6-8}$$

where

- $P^F_{i,|\mathbb{T}|,s}$ is the price of the asset $i \in A$ in the scenario $s \in S$ at the end of the simulated period $|\mathbb{T}|$ (for example, 5 or 20 days ahead);
- P^I_i is the price of the asset $i \in A$ at the last available historical date;
- Q_i is the quantity (in MM BBL[1]) of asset $i \in A$ that should be decided by the optimization model;
- R_s is the portfolio revenue in scenario $s \in S$;
- A is the set of available assets and S is set of simulated scenarios (ex: 1000 scenarios).

[1] MM BBL refers to one million barrels.

6.5 Risk Analysis and Backtest

Finally, we conclude our method by applying the backtesting techniques described in Chapter 5 in our portfolio. The backtesting is an important instrument to check for model adequacy in the risk context used in order to measure the number of violations on the estimated series and in the historical one.

Using the procedure described in section 6.3, we simulated the "out-of-sample" performance of the portfolio 20 day ahead ($\mathbb{T} = \{1, \ldots, 20\}$), in order to capture the uncertainty in the *commodity market*. Although the methodology described in Section 6.3 can be applied for a generic step forward period of time, it is important to mention that an accurate simulation is dependent of the number of steps. Therefore, for a long-term operation, a recalibration of the portfolio is recommended. With the proposed methodology, this recalibration can be easily made, with low computationally expenses. In this sense, in the next chapter, we present a set of case studies to discuss and validate the proposed methodology within different contexts.

7
Results

In this chapter, we will implement the method described in Chapter 6. This chapter is of utmost importance because it validates the proposed method. Each section will present a different Case Study and its particularities.

Investors pursue investments with high return and low risks. In order to achieve this objective, managers have to select a portfolio for their investors arranging the current wealth of investors over a set of available assets.

Risk, in a financial context, is a measure for uncertainty. So, *performance measures* and *risk measures* are instruments of estimating the worst-case scenarios, because they capture the downside and upside potential of the selected portfolio, while remaining consistent with utility maximization. In this chapter, we will compare various risk measures and their performance in a portfolio of commodities.

In the following sections, we will apply the method explained in Chapter 6 to different portfolios of commodities. Investing in actual physical commodities is the best option for producers and consumers of raw materials.

7.1 Case Study 1

In this case study, we consider a portfolio of 4 assets: *Brent, WTI, Petrobras* and *RBOB*[1]. The time period was January 2nd, 2007 until October 31st, 2013. This time period was chosen carefully because includes the 2008 crisis and the period after the 2008 crisis. The data was obtained from *Bloomberg*. Also, we will implement the method described in Chapter 6.

7.1.1 CVaR Optimization

In the first part of this subsection, we will apply PCA using only part of the data available from January 2nd, 2007 until August 10th, 2008. This period was chosen due to fact that is before the 2008 crisis.

Once we applied PCA to this subset of data, we were able to obtain the *scores* and the *loadings*. In possession of this results, we were able to choose the most relevant components of the score, that were able to expain 92% of the portfolio. In

[1] This is an hypothetical portfolio that is relevant for producers and consumers of raw materials.

order to calibrate the model applied to the obtained scores, different distributions were tested in the *scores*, namely: Gaussian, Student t and GARCH. In order to contrast those results, we used the AIC and BIC criteria and the best fit was the GARCH (1,1) model with t-innovations. After, we were able to simulate the returns and, consequently, the prices of each asset of the portfolio.

In order to decide the optimal allocations of our portfolio, we implemented the optimization (using the software *Xpress*) for the most intuitive risk measure: the *Conditional Value-at-Risk (CVaR)* [24]. The following allocations result at: Brent (20,3%), WTI (51,6%), Petrobras (10%) and RBOB (18,1%).

In possession of the allocations resultant from the maximization of the *CVaR*, we simulated 30 days ahead of the data used in the PCA and compared this result to the historical *CVaR* and *VaR* with a confidence level of 95%, considering a sliding window of 30 days. Figure 7.1 presents the *VaR* and *CVaR* (with a confidence level of 95%) results for 30 days of simulation using the optimal weights maximizing the *CVaR* and the comparison with the historical *VaR* and *CVaR*, considering a sliding window of 30 days:

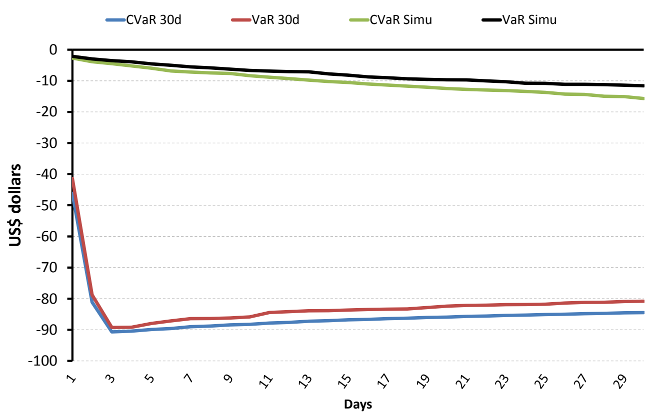

Fig. 7.1: Time evolution of the estimated measures (VaR and CVaR) along 30 days period from 1000 simulation and historical VaR and CVaR considering a sliding window of 30 days before 2008 crisis, using CVaR optimization.

As one can see, the results in Figure 7.1 were very inaccurate. The model was not able to forecast the 2008 crisis. The *CVaR* simulation predicted an approximate loss of 15 dollars for the investor and the *VaR* an approximate loss of 11 dollars. Furthermore, the historical *VaR* and *CVaR* had an approximate loss of 90 dollars.

This demonstrates the consequences of having a static and non-hedged portfolio.

We obtained an interesting result by changing the period for which PCA was applied in Figure 7.1. Due to fact that the PCA was applied in a different period of time, the most 3 relevant components explained 94% of the original portfolio. The new period chosen to repeat the whole method was the period from January 2nd, 2009 until November 1st, 2009. This period was carefully chosen to illustrate the performance of the method proposed after crisis.

Once again, the allocations were obtained by the maximization of the *CVaR*. We found different allocations to apply in the portfolio: Brent (23,1%), WTI (41,2%), Petrobras (15,5%) and RBOB (20,2%). The results are presented in Figure 7.2:

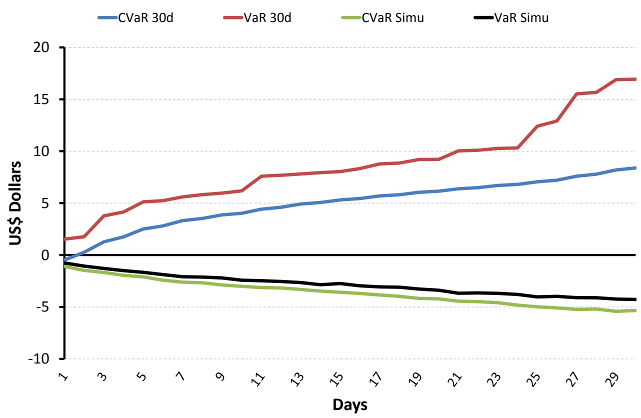

Fig. 7.2: Time evolution of the estimated measures (VaR and CVaR) along 30 days period from 1000 simulation and historical VaR and CVaR considering a sliding window of 30 days after 2008 crisis, using CVaR optimization.

Note that according to our convention, negative results correspond to losses and positive results correspond to gains. In red and blue, we consider a sliding window of 30 days for the calculus of the historical VaR and CVaR, respectively. In black and green, we can see the time evolution of the estimated measures (VaR and CVaR, both with a confidence level of 95%) along 30 days period from 1000 simulation.

Surprisingly, the results presented in Figure 7.2 were very coherent and satisfactory. Actually, the investor won in the historical VaR and CVaR and our simulation predicted a soft loss. This means that our model is more conservative than what really happened.

In the next subsection, we will repeat the same method presented above using the VaR maximization instead of CVaR Optimization.

7.1.2 VaR Optimization

A different setting is created when we decide to choose the portfolio allocations by maximizing the *Value-at-Risk (VaR)* (using again the software *Xpress*). We applied PCA using only part of the data available: January 2rd, 2007 until August 10th, 2008. The data was carefully chosen in order to represent the period before the 2008 crisis. We found the following allocations: Brent (10%), WTI (10%), Petrobras (70%) and RBOB (10%).

In possession of the allocations, we simulated 30 days ahead of the sample used in PCA and compared the results to the historical *VaR* and *CVaR* of the data, considering a sliding window of 30 days. Figure 7.3 presents *VaR* and *CVaR* (both with a confidence level of 95%) results for 30 days of simulation using the optimal weights maximizing the *VaR* and the comparison with the historical *VaR* and *CVaR*, also both with a confidence level of 95% considering a sliding window of 30 days:

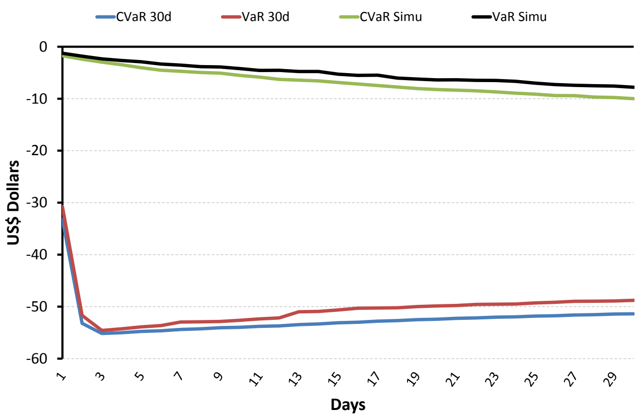

Fig. 7.3: Time evolution of the estimated measures (VaR and CVaR) along 30 days period from 1000 simulation and historical VaR and CVaR considering a sliding window of 30 days before 2008 crisis, using VaR optimization.

Once again, in Figure 7.3 the results were very inaccurate and the model proposed was not able to forecast the 2008 crisis. The *VaR* simulation predicted an approximate loss of 8 dollars for the investor and the *CVaR*, an approximate loss of 10 dollars. The historical *VaR* and *CVaR* had an approximate loss of 55 dollars.

Utilizing the same method applied above, a better result was obtained by changing the period which the PCA was applied in the last Figure 7.3. The new period chosen to repeat the whole process of the *VaR* and *CVaR* (both with a confidence level of 95%) simulation was the time period from January 2nd, 2009 until November 1st, 2009. This time period was chosen in order to represent the period after the 2008 crisis and how the method performs. We found the following allocations: Brent (11%), WTI (12%), Petrobras (65%) and RBOB (12%).

Once decided the weights by the maximization of *VaR*, we simulated 30 days ahead of the sample used in PCA and compared this result to the historical *VaR* and *CVaR* both with a confidence level of 95%, considering a sliding window of 30 days: As expected, the results presented in the Figure 7.4 were fairly coherent and satisfactory.

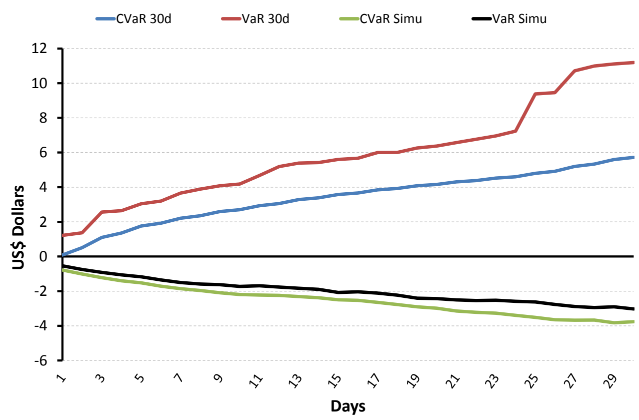

Fig. 7.4: Time evolution of the estimated measures (VaR and CVaR) along 30 days period from 1000 simulation and historical VaR and CVaR considering a sliding window of 30 days after 2008 crisis, using VaR optimization.

Note that according to our convention, negative results correspond to losses and positive results correspond to gains. In red and blue, we consider a sliding window of 30 days for the calculus of the historical VaR and CVaR, respectively. In black and green, we can see the time evolution of the estimated measures (VaR and CVaR) along 30 days period from 1000 simulation.

The subsequent subsection presents the out-of-sample of the VaR and the CVaR allocations.

7.1.3 Out-of-sample with CVaR optimization

In order to test how the optimal allocations would perform in the historical data, we present a case study that compares the out-of-sample with the historical *VaR* and *CVaR*. Those graphics were built to understand the crisis effects in the out-of-sample process.

Figure 7.5 represents the cumulative performance of the optimal allocations found in Subsection 7.1.1 (applying PCA in January 2th, 2007 until August, 10th, 2008). This out-of-sample, from August 11st, 2008 until October, 31st, 2013, illustrates a long-term portfolio:

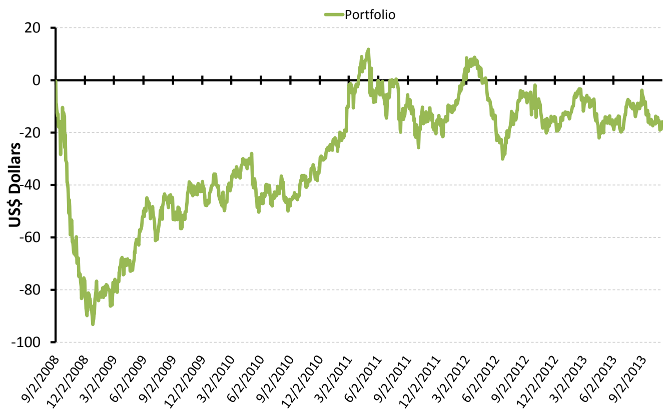

Fig. 7.5: Cumulative out-of-sample performance using CVaR optimization and out-of-sample adding the crisis period of 2008.

Figure 7.5 does not indicate a good performance of the portfolio. This is probably due to the fact that out-of-sample begins before 2008 crisis.

As an alternative, the Figure 7.6 represents different scenarios and allocations decided in Subsection 7.1.2, using PCA applied from January 2nd, 2009 until November 1st, 2009. Figure 7.6 represents this out-of-sample, from November 2nd, 2009 until October 31st, 2013, of a long-term portfolio:

Fig. 7.6: Cumulative out-of-sample performance using CVaR optimization and out-of-sample without the crisis period.

The result was very good if one compares to Figure 7.5. The investor was able to have a better return in the portfolio constructed after the crisis as we can see in the Figure 7.6.

7.1.4 Out-of-sample with VaR optimization

In Figure 7.7, we show the cumulative out-of-sample results with *VaR* optimization allocations. Figure 7.7 shows the results using the allocation decided before the 2008 crisis (applying PCA from January 2nd, 2007 until August 10th, 2008) in Subsection 7.1.1. This cumulative out-of-sample represents the period of August 11th, 2008 until October, 31th, 2013:

Fig. 7.7: VaR optimization and out-of-sample during the crisis period

As expected, Figure 7.7 presents a very bad performance. This happens probably because the out-of-sample begins before the 2008 crisis. We can see that this result is very similar to the result present in the Subsection 7.1.3 using the same period of time.

Finally, as an alternative, Figure 7.8 represents a different scenario and *VaR* optimal weights (after the 2008 crisis), using the PCA applied from November 2nd, 2009 until November 1st, 2009. Figure 7.8 illustrates the cumulative out-of-sample during November 2st, 2009 until October 31th, 2013:

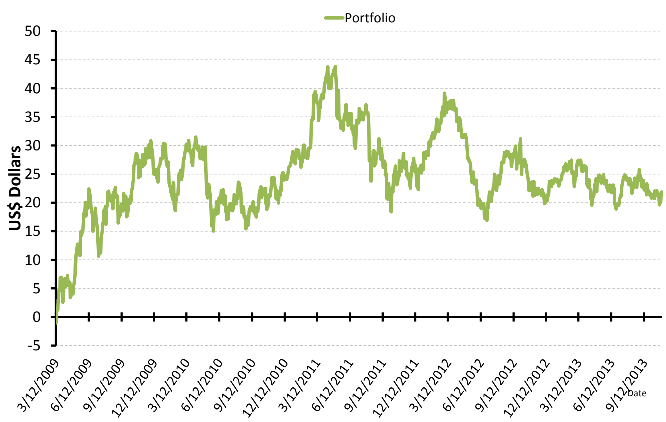

Fig. 7.8: Cumulative out-of-sample using VaR optimization after the crisis period

The result was very good if one compares to that of Figure 7.7. The portfolio constructed after the crisis had a better performance as we can see in Figure 7.8.

After substantial testing, one can conclude that the crisis of 2008 displays extreme events. The out-of-sample, done without the crisis of 2008, presented coherent results. If we include the 2008 historical data, we see a substantial loss due to the crisis.

7.2 Case Study 2

In this case study, the PCA had a relevant role in order to understand and preserve the correlation between the 12 chosen assets: *WTI, RBOB, Brent, Propane, Heating Oil, JET, ULSD, UNL, Naphta, FO 1%, FO 3,5% and Eurobob*[2]. It is important to say that the investor can go long or short in any of these assets. Due to the difficulty, the data available for all those assets was September 14th, 2011 until December 27th, 2013, resulting in 574 daily observations.

This second Case Study will be divided in 3 subsections: *VaR Optimization, CVaR Optimization* and *Omega Optimization*. In each subsection, we used the same method explained in Chapter 6: We first apply PCA to the original returns series (September 14th, 2011 until November 30th, 2013), then we chose the most relevant *Components* (in this particular case study, only 4 components were able to explained 94% of a portfolio of 12 assets) and finally, we will model individually the *Scores* of each of the chosen *Component* as a GARCH(1,1) model with t-innovations. After, we were able to simulate the returns (preserving the correlation that the assets hold to each other) and the prices of each asset of the portfolio.

In order to decide the optimal allocation of each asset, we implemented the optimization (using the software *Xpress*) for the following 3 risk measures: *VaR, CVaR* and *Omega*. In each risk measure, we optimize 2 portfolios: Optimal Static Portfolio, whose allocations are decided just once and the Optimal Dynamic Portfolio, whose allocations are recalibrated each 5 days.

Finally, we did a 20 days ahead simulation with the optimal allocations: contrasting the Optimal Static Portfolio and the Optimal Dynamic Portfolio (recalibrating the model each 5 days) with our *benchmark* Portfolio, which is composed by the same 12 assets presented above with static allocations that will be presented in

[2] This portfolio was carefully chosen due to its importance in the *commodity market*. Those assets are known as components and *benchmarks* of gasoline.

the next subsection.

7.2.1 VaR Optimization

In this Case Study, we will follow the method explained in the beginning of this section. In this subsection, the allocations were decided by the *VaR Optimization* in order to compare the performance of the Optimal Dynamic Portfolio and the Optimal Static Portfolio, having a fixed Portfolio as a *benchmark*.

Table 7.1 refers to the optimal allocations of the Optimal Dynamic Portfolio and its recalibration each 5 days during 4 weeks (20 days):

Assets	Optimal Dynamic Portfolio (MM BBL)			
	1st Calibration	2nd Calibration	3rd Calibration	4th Calibration
WTI	3	3	3	3
RBOB	3	3	3	3
BRENT	3	3	3	3
Propane	-0.34	0.75	0.075	-0.54
Heating Oil	-0.62	-0.55	0.60	-0.75
JET	-3	-3	-3	-3
ULSD	-3	-3	-3	-3
UNL	-2	-1.59	-3	-1.67
Naphta	2.54	1.5	0.27	1.9
FO 1% Oil	3	3	3	3
FO 3.5%	-3	-3	-3	-3
Eurobob	-1.7787	-3	-1.45	-2.89

Tab. 7.1: Optimal Dynamic Portfolio allocations using GARCH simulations and VaR Optimization.

We can notice that Table 7.1 shows a wide variability in the allocations each week. This is due to fact that the model incorporates more information about the market each time the model is recalibrated (every week for 20 days).

Also, according to the *VaR* optimization, we obtained the optimal allocation of the Optimal Static Portfolio for the 20 days ahead. Table 7.1 shows the optimal allocation of the Optimal Static Portfolio and the fixed allocation of the *benchmark* Portfolio:

Assets	Optimal Static Portfolio (MM BBL)	Benchmark Portfolio (MM BBL)
WTI	3	-3
RBOB	-2.50	0.3
BRENT	-3	-2
Propane	0.1	0.4
Heating Oil	3	0.4
JET	-3	0.2
ULSD	-0.50	0.4
UNL	3	0.3
Naphta	-2.24	0.2
FO 1% Oil	-3	-0.25
FO 3.5%	3	-0.25
Eurobob	3	0.1

Tab. 7.2: Optimal Static Portfolio allocations using GARCH simulations and VaR Optimization and benchmark portfolio.

In order to compare the performance of the three portfolios (Optimal Dynamic Portfolio, Optimal Static Portfolio and benchmark Portfolio), we present the cumulative out-of-sample given by the following Figure:

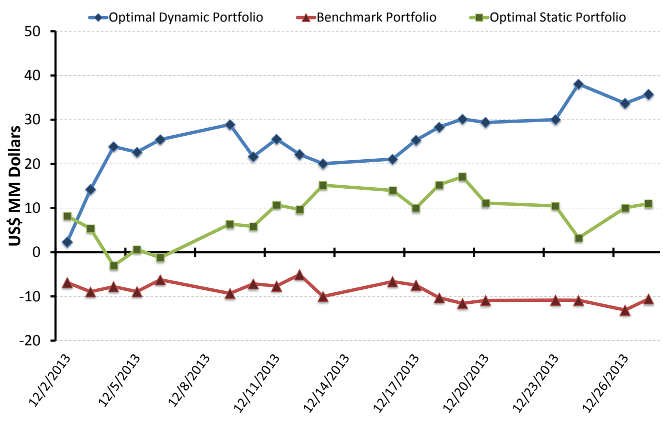

Fig. 7.9: Cumulative out-of-sample of Optimal Dynamic Portfolio, Optimal Static Portfolio and benchmark Portfolio for 20 days ahead.

We can see in Figure 7.9 that the Optimal Dynamic Portfolio has the best performance among the three Portfolios, followed by Optimal Static Portfolio and then, our benchmark Portfolio. This means that the recalibration at each 5 days (Optimal Dynamic Portfolio) is effective in the final results.

The Optimal Dynamic Portfolio had a better performance because, in each recalibration, the system includes more 5 days of the data in the optimization. So, the simulations try to follow the tendencies of the market.

7.2.2 CVaR Optimization

In this subsection, we will use the same data described in the beginning of this section and the same time period.

In order to discover the optimal allocations of the Optimal Dynamic Portfolio, we apply the method explained in the beginning of this section. Table 7.3 refers to the optimal allocations of the Optimal Dynamic Portfolio and its recalibration each 5 days (each week) during 20 days using the *CVaR optimization*:

Assets	Optimal Dynamic Portfolio (MM BBL)			
	1st Calibration	2nd Calibration	3rd Calibration	4th Calibration
WTI	3	3	3	3
RBOB	3	3	3	3
BRENT	-1.24	3	3	3
Propane	-3	0.67	0.33	-0.35
Heating Oil	-2	-1.77	-0.31	-1.1
JET	-3	-3	-3	-3
ULSD	-2.5	-3	-3	-3
UNL	3	-0.63	-1.8	-1.4
Naphta	-1.79	1.8	1.9	1.2
FO 1% Oil	2.38	3	3	3
FO 3.5%	-3	-3	-3	-3
Eurobob	1.9	-3	-3	-2.9

Tab. 7.3: Optimal Dynamic Portfolio allocations using GARCH simulations and CVaR optimization.

The allocations illustrated in Table 7.3 show some variations in the recalibration. The result of the optimization is different every week due to the fact that the inputs are different, i.e., the model incorporates more market information.

Also, according to the *CVaR* optimization, we obtained the Optimal Static Portfolio for the 20 days ahead (using the same allocations during this period). Table 7.4 shows the allocation of the Optimal Static Portfolio and the fixed allocation of our *benchmark* Portfolio:

Assets	Optimal Static Portfolio (MM BBL)	Benchmark Portfolio (MM BBL)
WTI	3	-3
RBOB	-2.72	0.3
BRENT	-3	-2
Propane	0.11	0.4
Heating Oil	3	0.4
JET	-3	0.2
ULSD	-0.58	0.4
UNL	3	0.3
Naphta	-2.17	0.2
FO 1% Oil	-3	-0.25
FO 3.5%	3	-0.25
Eurobob	3	0.1

Tab. 7.4: Optimal Static Portfolio allocations using GARCH simulations and CVaR Optimization and the benchmark Portfolio.

In order to compare the performance of the Optimal Dynamic Portfolio, Optimal Static Portfolio and the benchmark Portfolio, we did a cumulative "out-of-sample" test 20 days ahead. Figure 7.10 shows the results

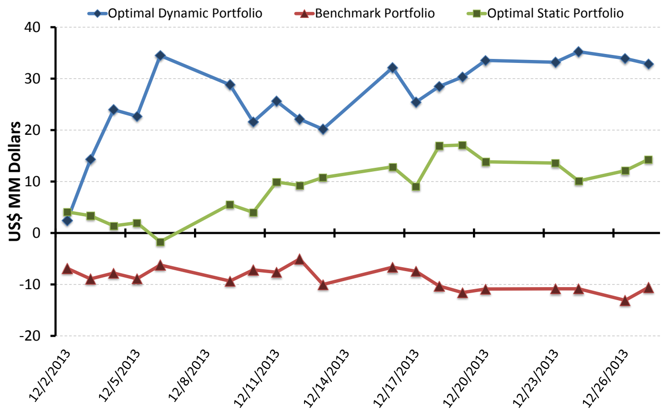

Fig. 7.10: Cumulative out-of-Sample example with Optimal Dynamic Portfolio, Optimal Static Portfolio and benchmark Portfolio.

It is interesting to notice the results presented in Figure 7.10: the Optimal Dynamic Portfolio had the best performance, followed by the Optimal Static Portfolio, and finally, the benchmark Portfolio in the cumulative "out-of-sample" test 20 days ahead. This results happen due to the fact that the Optimal Dynamic Portfolio has access to more historical data (in each recalibration) then the Optimal Static Portfolio and the benchmark Portfolio. In this way, the price simulation of the Optimal Dynamic Portfolio is more reliable and coherent because of its reduced uncertainty.

7.2.3 Omega Optimization

Once again, we will use the same data described in the beginning of this section 7.2 and the same period of time, so that we can compare the performance of each optimized risk measure.

Finally, we will optimize the allocations using the Linear Programming of *Omega Ratio*, presented in Chapter 3. Table 7.5 refers to the optimal allocations of the Optimal Dynamic Portfolio and its recalibration each 5 days (week) for 20 days (4 weeks):

Assets	Optimal Dynamic Portfolio (MM BBL)			
	1st Calibration	2nd Calibration	3rd Calibration	4th Calibration
WTI	3	3	3	3
RBOB	3	3	3	3
BRENT	3	3	3	3
Propane	-0.41	0.74	0.33	-0.35
Heating Oil	-1.74	-1.9	-0.41	-0.8
JET	-3	-3	-3	-3
ULSD	-3	-3	-3	-3
UNL	-1.19	-0.73	-1.75	-1.47
Naphta	2.28	1.9	2.4	1.5
FO 1% Oil	3	3	3	3
FO 3.5%	-3	-3	-3	-3
Eurobob	-3	-3	-3	-2.4

Tab. 7.5: Optimal Dynamic Portfolio allocations using GARCH simulations and Omega Optimization.

The allocations illustrated in Table 7.5 show the variations in the recalibration proposed by the method. The result of the optimization is different every week due to the fact that the inputs are different, i.e., the model incorporates more market information.

Also, according to the *Omega* optimization, we obtained the optimal allocation of the Optimal Static Portfolio A for the 20 days ahead. Table 7.6 shows the optimal allocation of the Optimal Static Portfolio and the fixed allocation of the *benchmark* Portfolio:

Assets	Optimal Static Portfolio (MM BBL)	Fixed Portfolio (MM BBL)
WTI	3	-3
RBOB	-1.9	0.3
BRENT	-3	-2
Propane	-0.05	0.4
Heating Oil	3	0.4
JET	-2.45	0.2
ULSD	-1.27	0.4
UNL	2.75	0.3
Naphta	-1.5	0.2
FO 1% Oil	-3	-0.25
FO 3.5%	3	-0.25
Eurobob	3	0.1

Tab. 7.6: Optimal Static Portfolio allocations using GARCH simulations and Omega Optimization and benchmark Portfolio.

Aiming to compare the results of each Portfolio, we did a cumulative "out-of-sample" test 20 days ahead, using the allocations presented above in Table 7.6 and 7.6. Figure 7.11 shows the results

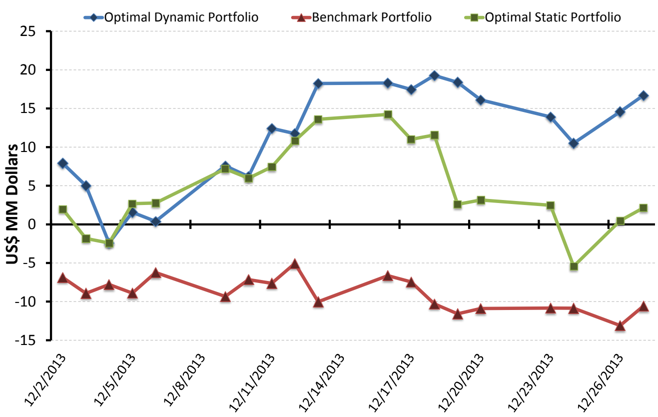

Fig. 7.11: Cumulative out-of-Sample example with Optimal Dynamic Portfolio, Optimal Static Portfolio and benchmark Portfolio.

Indeed, one can see that the Optimal Dynamic Portfolio, once again, had the best performance among the compared Portfolios. It is due to the fact, that Optimal Dynamic Portfolio was recalibrated each 5 days and the optimization was able to capture the tendencies of the commodities markets.

This result was very satisfactory because both of our Optimal Portfolios has outperformed the benchmark Portfolio. This means that the prices simulations were reliable and capable of extract the tendencies of the market.

7.3 Case Study 3

In this Case Study, we will apply the method described in Chapter 6. This section will be divided in 3 subsections. Each subsection represents the risk measure optimized *VaR, CVaR* and *Omega Ratio*.

Once again, we will use the same period of time (September 14th, 2011 until December 27th, 2013) and the same data described in the beginning of Section 7.2: the 12 chosen assets (*WTI, RBOB, Brent, Propane, Heating Oil, JET, ULSD, UNL, Naphta, FO 1%, FO 3,5%* and *Eurobob*). The frequency of data was daily. The same portfolio was chosen so that we could compare the performance of each optimized risk measure. In addition, it is important to say that the investor can go long or short in any of these assets.

The method of Case Study 3 consists in applying the PCA directly in the original returns (September 14th, 2011 until November 30th, 2013) in order to potentially decrease the portfolio dimension. The main difference between Case Study 2 and Case Study 3 is that in Case Study 3, the *Scores* were modeled as *Geometric Brownian Motion* instead of *GARCH*, as used in Case Study 2.

Once decided the most relevant *Components* of the result of PCA, we simulated the simulated prices by using the method presented by Equation (6-7). Thus, the correlations that the assets hold to each other and the *Scores* volatility will be preserved in the simulated prices.

Aiming to decide the optimal allocation of each asset, we optimized (using the software *Xpress*) the following 3 risk measure: *VaR, CVaR* and *Omega*. In each risk measure, we optimize 2 portfolios: Optimal Static Portfolio, whose allocations are decided just once and the Optimal Dynamic Portfolio, whose allocations are recalibrated each 5 days.

Finally, we did a 20 days ahead simulation with the optimal allocations: contrasting the Optimal Static Portfolio and the Optimal Dynamic Portfolio (recalibrating the model each 5 days) with our *benchmark* Portfolio, composed by the same 12 assets and static allocations.

7.3.1 VaR Optimization

We will follow the method explained in the beginning of this section. According to the results of the *VaR Optimization*, we found the optimal allocations of the Optimal Dynamic Portfolio. Table 7.7 refers to the optimal allocations of the Optimal Dynamic Portfolio and its recalibration each 5 days (week) for 20 days (4 weeks):

Assets	Optimal Dynamic Portfolio (MM BBL)			
	1st Calibration	2nd Calibration	3rd Calibration	4th Calibration
WTI	0.26	0.19	1.38	-0.28
RBOB	0.04	-0.10	-0.18	0.32
BRENT	-0.47	0.39	-0.11	0.29
Propane	0.1	-0.08	0.02	0.24
Heating Oil	-0.06	-0.65	0.63	-0.03
JET	-0.17	0.48	-0.12	-0.22
ULSD	-0.69	0.20	0.79	0.16
UNL	-0.92	-0.05	0.01	0.26
Naphta	0.08	-0.21	0.52	0.58
FO 1% Oil	0.10	-0.06	-0.10	0.11
FO 3.5%	-0.11	-0.14	-2.11	-0.37
Eurobob	-0.06	-0.17	0.02	0.05

Tab. 7.7: Optimal Dynamic Portfolio allocations using GBM simulations and VaR Optimization.

The allocations illustrated in Table 7.7 show the variations in the recalibration proposed by the method. The result of the optimization is slightly different every week due to the fact that the inputs are different, i.e., the model incorporates more market information.

Also, according to the *VaR* optimization, we were able to obtain the allocation of the Optimal Static Portfolio for the 20 days ahead. Table 7.8 shows the optimal allocation of the Optimal Static Portfolio and the fixed allocation of the *benchmark* Portfolio:

Assets	Optimal Static Portfolio (MM BBL)	Benchmark Portfolio (MM BBL)
WTI	0.20	-3
RBOB	0.03	0.3
BRENT	1.49	-2
Propane	0.89	0.4
Heating Oil	-0.11	0.4
JET	-0.35	0.2
ULSD	-0.24	0.4
UNL	-0.08	0.3
Naphta	-0.05	0.2
FO 1% Oil	-0.17	-0.25
FO 3.5%	-0.24	-0.25
Eurobob	0.08	0.1

Tab. 7.8: Optimal Static Portfolio allocations using GBM simulations and VaR Optimization and Benchmark Portfolio.

The allocations presented in Table 7.7 were decided by the implementation of maximization of the *VaR*. In possession of this allocations, we did a cumulative "out-of-sample" test 20 days ahead, in order to compare the performance of the three presented Portfolios (Optimal Dynamic Portfolio, Optimal Static Portfolio and benchmark Portfolio). Figure 7.12 shows the results:

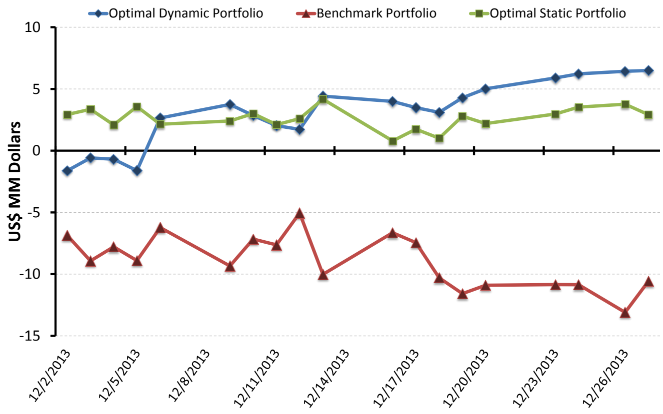

Fig. 7.12: Cumulative out-of-Sample example with Optimal Dynamic Portfolio, Optimal Static Portfolio and benchmark Portfolio.

In this cumulative out-of-sample, we can notice that the Optimal Dynamic Portfolio outperformed the Optimal Static Portfolio and benchmark Portfolio. This is due to the fact that the Optimal Dynamic Portfolio recalibrate its allocations every 5 days, identifying correctly the tendencies of the market.

7.3.2 CVaR Optimization

In this section, we will optimize the allocations by using the Linear Programing of CVaR. But first, we will apply the method already explained in the beginning of this section.

Table 7.9 refers to the optimal allocations of the Optimal Dynamic Portfolio and its recalibration each 5 days (week) for 20 days (4 weeks):

Assets	Optimal Dynamic Portfolio (MM BBL)			
	1st Calibration	2nd Calibration	3rd Calibration	4th Calibration
WTI	-0.24	1.12	1.25	0.93
RBOB	-0.14	0.58	0.41	0.24
BRENT	3	-0.37	-0.13	0.32
Propane	0.15	0.18	0.05	0.37
Heating Oil	-0.40	-0.35	0.14	-0.16
JET	0.75	-0.19	-0.11	0.56
ULSD	-2.5	-0.04	0.17	0.18
UNL	-0.80	0.13	0.28	0.11
Naphta	2.89	-0.71	0.07	-0.38
FO 1% Oil	-2.69	0.15	-1.3	0.10
FO 3.5%	0.63	0.52	-0.24	-0.14
Eurobob	0.27	0.19	-0.06	-0.05

Tab. 7.9: Optimal Dynamic Portfolio allocations using GARCH simulations and CVaR Optimization.

Table 7.9 illustrates the results of the recalibration proposed by the method. The result of the optimization is slightly different every week due to the fact that the inputs are different, i.e., the model incorporates more market information in the simulations.

Also, according to the *CVaR* optimization, we were able to obtain the optimal allocation of the Optimal Static Portfolio for the 20 days ahead. Table 7.10 shows the allocations of the Optimal Static Portfolio and the fixed allocation of the *benchmark* Portfolio:

Assets	Optimal Static Portfolio (MM BBL)	Benchmark Portfolio (MM BBL)
WTI	0.16	-3
RBOB	0.2	0.3
BRENT	-0.37	-2
Propane	0.13	0.4
Heating Oil	0.67	0.4
JET	-0.05	0.2
ULSD	0.61	0.4
UNL	-0.29	0.3
Naphta	0.51	0.2
FO 1% Oil	-0.04	-0.25
FO 3.5%	0.27	-0.25
Eurobob	0.03	0.1

Tab. 7.10: Optimal Static Portfolio allocations using GBM simulations and CVaR Optimization and benchmark Portfolio.

One way of comparing the performance among the Optimal Dynamic Portfolio, Optimal Static Portfolio and the benchmark Portfolio is to perform a "out-of-sample" test 20 days ahead. Figure 7.13 shows the results of this example

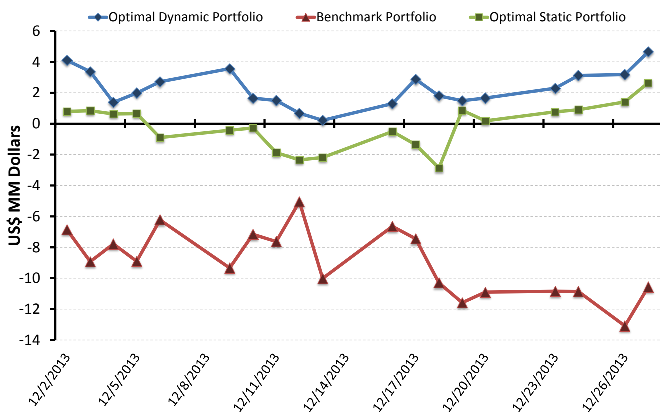

Fig. 7.13: Cumulative out-of-Sample example with Optimal Dynamic Portfolio, Optimal Static Portfolio and the benchmark Portfolio using GBM simulations.

The result that Figure 7.13 presents shows that the Optimal Dynamic Portfolio had a better performance when compared Optimal Static Portfolio and the benchmark Portfolio. This result was expected due to the fact that the Optimal Dynamic Portfolio have access to more historical data (in each relaibration) than the Optimal Static Portfolio and the benchmark Portfolio, so the simulated prices of the Optimal Dynamic Portfolio are more align with the tendencies of the market.

7.3.3 Omega Optimization

In this subsection, as explained in Section 7.3, the *Scores* will be modeled as *GBM*. Thus, we will optimize the allocations using the Linear Programming of *Omega Ratio*, already presented in Chapter 3. So, Table 7.11 illustrates the result of the optimal allocations of Optimal Portfolio A:

Table 7.11 refers to the optimal allocations of the Optimal Dynamic Portfolio and its recalibration each 5 days:

Assets	Optimal Dynamic Portfolio (MM BBL)			
	1st Calibration	2nd Calibration	3rd Calibration	4th Calibration
WTI	-0.18	1.47	2.31	0.28
RBOB	0.27	0.29	-0.03	0.13
BRENT	-1.61	-3	-2.69	3
Propane	0.23	0.27	0.77	0.21
Heating Oil	0.67	-0.28	0.33	0.05
JET	0.03	1.34	-0.39	0.38
ULSD	-1.31	-0.35	3	-0.44
UNL	0.08	-0.19	0.27	0.84
Naphta	1.19	-0.51	1.44	-0.29
FO 1% Oil	-0.79	1.10	-1.79	0.98
FO 3.5%	-3	1.04	-2.65	-0.82
Eurobob	0.16	0.05	0.14	0.34

Tab. 7.11: Optimal Dynamic Portfolio allocations using GBM simulations and Omega Optimization.

Table 7.11 illustrates the results of the recalibration proposed by the method. The result of the optimization is slightly different every week due to the fact that the inputs are different, i.e., the model incorporates more market information in the simulations.

According to the *Omega* optimization, we also obtained the allocation of the Optimal Static Portfolio for the 20 days ahead. Table 7.12 shows the allocation of the Optimal Static Portfolio and the fixed allocation of the *benchmark* Portfolio:

Assets	Optimal Static Portfolio (MM BBL)	Benchmark Portfolio (MM BBL)
WTI	0.8	-3
RBOB	-0.78	0.3
BRENT	-2.67	-2
Propane	0.84	0.4
Heating Oil	0.55	0.4
JET	-1.17	0.2
ULSD	-0.23	0.4
UNL	0.17	0.3
Naphta	-0.04	0.2
FO 1% Oil	-1.01	-0.25
FO 3.5%	3	-0.25
Eurobob	0.58	0.1

Tab. 7.12: Optimal Static Portfolio allocations using GBM simulations and Omega Optimization and Benchmark Portfolio.

Then, we did a "out-of-sample" test 20 days ahead in order to compare the performance of the Optimal Dynamic Portfolio, Optimal Static Portfolio and of the Benchmark Portfolio. Figure 7.14 shows the results

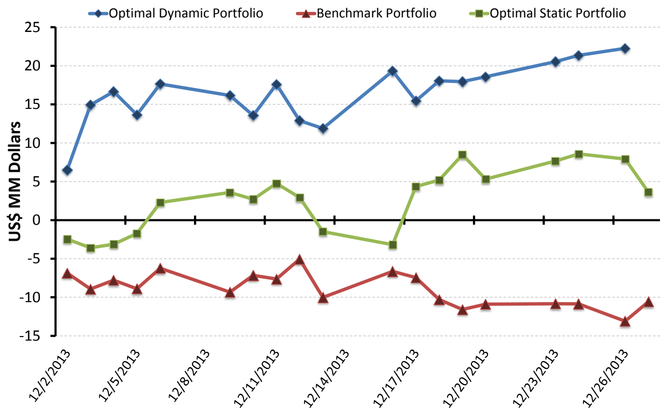

Fig. 7.14: Cumulative out-of-Sample example with Optimal Dynamic Portfolio, Optimal Static Portfolio and of the Benchmark Portfolio.

In Figure 7.14, we can see that, once again, the Optimal Dynamic Portfolio outperforms the Optimal Static Portfolio and the Benchmark Portfolio.

8
Conclusion

This work developed a numerical method to determine how to treat the correlation between the assets, simulation of prices and the optimization of a portfolio, maximizing various risk measures, such as *VaR, CVaR* and *Omega*. This methodology was illustrated in three case studies, presented in Chapter 7.

One of the main achievements of this work was to implement, as a Linear Program, the recently proposed performance measure *Omega Ratio* and the implementation of maximization of this measure.

Given the uncertainty present in *commodities markets*, an investor usually wants to protect his assets from potential losses. So, the optimization of risk measures, such as *VaR, CVaR* and *Omega*, can describe a range of future possible scenarios and their intrinsic risk.

In Case Study 1, we applied the method described in Chapter 6 in 4 assets portfolio. We applied this method in different periods of the data available in order to understand the impact of the 2008 crisis in our results. We modeled individually the scores of the most relevant components as *GARCH* processes. In each subsection, we optimized the allocations using the CVaR and the VaR maximization. The results, when applying PCA from January 2nd, 2007 until August 10th, 2008 (before the 2008 crisis) were surprising because our method was not able to predict the crisis and also the performance of the cumulative out-of-sample during August 11th, 2008 until October 31th, 2013 was very inaccurate.

To contrast the results with a after crisis period, we applied PCA on the original returns in different period of time, after 2008 crisis (January 2nd, 2009 until November 1st, 2009). Once again, we optimized the allocations using the CVaR and the VaR maximization. Our simulation was more conservative then what really happened. Also, the cumulative out-of-sample results done from November 2nd, 2009 until October 31th, 2013 were fairly satisfactory because were profitable.

Case Study 1 alerts to the consequences of the choice of a static and non-hedged portfolio. The Case Study 1 is different from Case 2 and 3, because it aims to contrast the results of the proposed method before and after the 2008 crisis.

In Case Study 2, we treated the daily log returns and then, applied Principal Component Analysis (PCA) method to the original returns, in order to (potentially) decrease the dimensionality of the problem. Once the most relevant components were found, we modeled individually the scores of the most relevant components as *GARCH* processes. After that, we were able to simulate the prices, preserving the correlations that the assets holds to each other.

Once the prices were simulated, we optimized the portfolio, maximizing the VaR, CVaR and Omega, in each subsection. In possession of these optimal allocations, we were able to perform a cumulative out-of-sample 20 days ahead. We optimized two portfolios: Optimal Dynamic Portfolio (recalibrated each 5 days for 20 days) and Optimal Static Portfolio (using fixed optimized allocations).

The results presented in Section 7.2 were coherent and better than expected. Our Optimal Dynamic Portfolio and Optimal Static Portfolio had a better performance than the benchmark Portfolio in the 20 days ahead out-of-sample in all subsections.

Additionally, in Case 3, we presented a different approach. The method consisted in applying Principal Component Analysis method to the original returns and, based on that, choosing the most relevant components. The difference between Cases 2 and 3 is that in Case 3, the scores were modeled as Geometric Brownian Motion process, whereas in Case 2 were modeled as a GARCH process. Our major concern was to simulate prices preserving the correlations among and their volatilities.

Each subsection of Case 3 represents the optimization of the VaR, CVaR and Omega in order to solve the optimal allocation problem. In possession of those allocations, the 20 day ahead cumulative out-of-sample of the portfolio.

The results are illustrated in Section 7.3. Once again, our Optimal Dynamic Portfolio and Optimal Static Portfolio outperformed the benchmark Portfolio in all Cases in the Case 3.

Since Sections 7.2 and 7.3 used the same Portfolio (12 assets) and the same time period, we were able to compare their results. In Case 2, the results of VaR Optimization had a better performance in the cumulative out-of-sample.

In a relative comparison between the results of Sections 7.2 and 7.3, we can say that the CVaR Optimization in Case 2, whose Scores were modeled as GARCH, presented higher returns in the 20 day ahead out-of-sample.

Finally, in Case 3, the Omega Optimization with GBM simulations results were more conservative and had a lower return than the Omega Optimization with GARCH simulations, in Case 2.

In conclusion, we cannot judge the best model or optimization, using GARCH or GBM, because both of them present good features. This decision has to be made according to the the investor’s objective to validate such choice.

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