Pontifícia Universidade Católica
do Rio de Janeiro

João Vitor Medeiros Domingos

A Priori Estimates with Application to Mean-Field Games

Dissertação de Mestrado

Dissertation presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática.

Advisor: Prof. Edgard Pimentel

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Abstract

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The mean-field games framework was developed to study problems with an infinite number of rational players in competition, which could be applied in many problems. The formalized study of these problems has begun, in the mathematical community by Lasry and Lions, and beside them, but independently close to the same time in the engineering community by P. Caines, Minyi Huang, and Roland Malhamé. Since these seminal contributions, the research in mean-field games has grown exponentially, and in this work we present a regularity to a case of mean-field games using particulars techniques.

In this work, we study time-dependent mean-field games in the subquadratic case, that is, mean-field games, which are written as a system of a Hamilton–Jacobi equation and a transport or Fokker–Planck equation, where The Hamiltonian presented on the Hamilton–Jacobi equation has a subquadratic growth. We begin by assuming ten assumptions, and then, under these assumptions derive Lipschitz regularity of the system.

Keywords

Assumptions; First and Second Order Estimates; Regularity for the Hamilton-Jacobi equation; Regularity for the Fokker-Plank equation; Sobolev Regularity; Lipschitz Regularity.

Resumo

Medeiros Domingos, João Vitor; Pimentel, Edgard. **Estimativas a priori e jogos de campo médio**. Rio de Janeiro, 2019. 55p. Dissertação de Mestrado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

A estrutura dos mean-filed games foi desenvolvida com o intuito de estudar problemas com um infinito número de jogadores em algum tipo de competição, ao qual pode ser aplicado em diversos problemas. O estudo formalizado desses problemas começou, na comunidade matemática com Lasry and Lions, e mais ou menos na mesma época, porém independentemente, na comunidade de engenharia por P. Caines, Minyi Huang, and Roland Malhamé. Desde então a pesquisa nos mean-field games cresceu exponencialmente, e nesse trabalho apresentaremos regularidade para um caso de mean-field games utilizando tecnicas particulares.

Nesse trabalho, estudamos time-dependent mean-field games no caso subquadrático, isto é, mean-field games, o qual é escrito como um sistema de duas equações, uma equação de Hamilton-Jacobi e uma equação do transporte ou uma equação de Fokker-Plank, em que o Hamiltoniano na equação de Hamilton-Jacobi possui um crescimento subquadratico. Começamos em assumir dez suposições, e então sob os mesmos deduzir regularidade Lipschitz para o sistema.

Palavras-chave

Suposições; Estimativas de Primeira e Segunda Ordem; Regularidade para a Equação de Hamilton-Jacobi; Regularidade para a Equação de Fokker-Plank; Regularidade Sobolev; Regularidade Lipschitz.

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O sábio nunca diz tudo o que pensa, mas pensa sempre tudo o que diz.
Aristóteles

1
Introduction

1.1
Mean-Field Games

The mean-field games formalism is a class of methods developed in series of seminal papers by J.-M. Lasry, P.-L. Lions [10]-[14] and M.Huang, R. Malhame and P. Caines [8, 9] which aims at understanding differential games with infinitely many indistinguishable players, in other words, differential games with a large population of rational players. These agents have preferences not only about their state (e.g., wealth, capital) but also on the distribution of the remaining individuals in the population. Mean-field games theory studies generalized Nash equilibrium for these systems. Usually, these models are characterized by a pair of coupled partial differential equations, known as a transport equation or Fokker-Plank equation for the distribution of the players and a Hamilton-Jacobi equation.

One of the most important research direction in the theory of mean-field games concerns the study of the existence and regularity of solutions. Well-posedness in the class of smooth solutions was explored, both in the stationary and in the time-dependent setting. A priori estimates are a fundamental ingredient for the pursuing of well-posedness (nonlinear) partial differential equations. This is the content of this work.

1.2
Hamilton-Jacobi and Fokker-Plank Equation

In this section, we discuss several notions from the setting of deterministic optimal control to the stochastic setting. We start by addressing the Hamilton-Jacobi equation.

1.2.1

Hamilton-Jacobi Equation

Consider a single player whose state is determined by a point $x \in R^d$.This agent can change its state by applying a control $v \in R^d$. However, the players are subject to independent external and random forces that are modeled by a white noise. In this simplified model, the trajectory of the player is given by the stochastic differential equation (SDE)

$$\begin{cases} dx_t = v_t dt + \sigma dW_t \\ x_{t_0} = x, \end{cases} \tag{1.1}$$

where v is a progressively measurable control.

Consider a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. By selecting the control v in a progressively measurable way, the player seeks to maximize a functional cost given by

$$J(v, x; t) = \mathbb{E}^x \left[\int_t^T L(x_s, v_s; m) ds + \Psi(x_T) \right], \tag{1.2}$$

where m represents a quantity to be made precise later. In 1.2 $\mathbb{E}^x$ denotes the expectation operator, given that $x_t = x$. furthermore $\Psi : \mathbb{R}^d \to \mathbb{R}$, is the terminal cost of the system.

The Legendre transform of L is

$$H(x, p; m) = \sup_{v \in \mathbb{R}^d} (p \cdot v + L(x, v; m)). \tag{1.3}$$

We are interested in the value function of this problem, u, which is determined by

$$u(x, t) = \sup_v J(x, v; t).$$

1.2.2
Fokker-Plank Equation

In this section, we examine the Fokker-Planck equation. Consider a population of players whose state is $x \in \mathbb{R}^d$. Assume further that the state of each agent in the population is governed by the stochastic differential equation in (1.1). Under the assumption of uncorrelated noise, the evolution of the population's density is determined by a Fokker-Planck equation. To discuss the derivation of this equation, we depend once more on the notion of infinitesimal generator of a (Markov) process. We refer the reader interested in stochastic analysis to [1] and [?]

Let A be the generator of a Markov process x_t. The formal adjoint of A, denoted by A^*, acts on functions in a suitable regularity class and is determined by the identity

$$\int_{\mathbb{R}^d} \phi(x) A f(x) dx = \int_{\mathbb{R}^d} f(x) A^* \phi(x),$$

for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Example 1.2.1 (Markov diffusion) *The infinitesimal generator of a Markov diffusion is*

$$A^v[f](x,t) = \frac{\partial}{\partial t} f(x,t) + h \cdot f_x(x,t) + \frac{\operatorname{Tr} \sigma^T \sigma D_x^2 f}{2}.$$

Therefore A^* is given by

$$(A^v)^*[m] = -\frac{\partial}{\partial t} m - \operatorname{div}(hm) + \frac{\left((\sigma^T\sigma)_{i,j} m\right)_{x_i x_j}}{2}$$

A fundamental result is given an initial configuration m_0, is described by the equation:

$$\begin{cases} A^*[m](x,t) = 0 \\ m(x,t_0) = m_0(x). \end{cases} \tag{1.4}$$

The Example 1.2.1 build upon (1.4) yields the Fokker-Planck equation

$$m_t(x,t) + \operatorname{div}(hm(x,t)) = \frac{\left((\sigma^T\sigma)_{i,j} m\right)_{x_i x_j}}{2}$$

1.3
Time-Dependent Mean-Field Games

In this work, we explored the regularity of the subquadratic case.

A model studied here is the system

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g(m) \\ m_t - \operatorname{div}(D_p H m) = \Delta m. \end{cases} \tag{1.5}$$

Where, H and g satisfy specific conditions as detailed in Section 1.4 We can coupled the system above with the boundary conditions, knows as, initial-terminal boundary conditions:

$$\begin{cases} u(x, T) = u_0(x) \\ m(x, 0) = m_0(x), \end{cases} \tag{1.6}$$

where $T > 0$ is a fixed terminal instant. We will consider in this work spatially periodic solutions. That is, u and m are regarded as functions with domain $\mathbb{T}^d \times [0, T]$, where $\mathbb{T}^d$ is the d-dimensional torus. The main goal of this work is to obtain conditions under which existence of solution to (1.5) under the initial-terminal conditions (1.6) can be established. We considered, in this paper, also a model non-linearity $g(m) = m^\alpha$, and improve and extend for Hamiltonians with subquadratic growth. So in this work we are interest to show the Lipschitz regularity for u.

In order to prove our goal, we consider a regularization of (1.5) by replacing $g(m)$ by the nonlocal operator

$$g_\epsilon(m) = \eta_\epsilon * g(\eta_\epsilon * m) = \int_{\mathbb{T}^d} \eta_\epsilon(x) \cdot g\left(\int_{\mathbb{T}^d} \eta_\epsilon(z) \cdot m(y - x - z) dz \right) dx,$$

where η_ϵ is a standard mollifying kernel, which in particular is symmetric. This yields the system

$$\begin{cases} -u_t^\epsilon + H(x, D_x u^\epsilon) = \Delta u^\epsilon + g_\epsilon(m^\epsilon) & \text{in} \quad \mathbb{T}^d \times (0, T) \\ m_t^\epsilon - \operatorname{div}(D_p H m^\epsilon) = \Delta m^\epsilon & \text{in} \quad \mathbb{T}^d \times (0, T). \end{cases} \tag{1.7}$$

We use the convention $g_0 = g$. The proof proceeds by establishing a new class of polynomial estimates for m^ϵ, which combined with upper bounds for u^ϵ.

Under the ten specific Assumptions that we are going to see later, follow our main results.

Teorema 1.1 *Let* (u^ϵ, m^ϵ) *be a solution of (1.7). Suppose* $m^\epsilon \in L^\infty([0, T]; L^{\beta_0}(\mathbb{T}^d))$, $\beta_0 \geq 1$. *Assume that* $p > \frac{d}{2}$, *let* q *be the conjugate* *exponent and* $r = \frac{1}{\kappa}$, *where*

$$\kappa = \frac{d + 2q - dq}{q[(\theta - 1)d + 2]}. \tag{1.8}$$

Then,

$$\int_{\mathbb{T}^d} (m^\epsilon)^{\beta_n}(x,\tau)dx \leq C + C \left\| |D_pH|^2 \right\|^{r_n}_{L^r([0,T];L^p(\mathbb{T}^d))}, \tag{1.9}$$

where

$$r_n = r\frac{\theta^n - 1}{\theta - 1}, \tag{1.10}$$

$\theta > 1$, $n \in \mathbb{N}$ *and* $\beta_n = \theta^n \beta_0$.

The proof of Theorem 1.1 is presented in Chapter 3. The key upper bounds for u^ϵ are given by:

Lemma 1.3.1 *Let* (u^ϵ, m^ϵ) *be a solution of (1.7) and assume that A1-A7 are in force. Let* $a, b > 1$ *be such that*

$$\frac{d}{2} < \frac{b(a-1)}{a}.$$

Then these exists $C > 0$ *such that*

$$\|u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} \leq C + C\|g_\epsilon(m^\epsilon)\|_{L^a([0,T];L^b(\mathbb{T}^d))}.$$

Lemma 1.3.1 is proved in Chapter 4. Using Gagliardo-Nirenberg interpolation Theorem we get:

Theorem 1.3.2 *Let* (u^ϵ, m^ϵ) *be a solution of (1.7) and assume that A1-9 are in force. For* $1 < p, r < \infty$ *there are* $c, C > 0$ *such that*

$$\|D^2u^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq c\|g_\epsilon(m^\epsilon)\|_{L^r([0,T];L^p(\mathbb{T}^d))} + c\|u^\epsilon\|^{\frac{\gamma}{2-\gamma}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + C. \tag{1.11}$$

The proof of Theorem 1.3.2 is presented in Chapter 5.

1.4 Main Assumptions

In this Section we present the main assumptions used throughout this work.

1.4.1
Assumptions

A1. The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $d > 2$, is smooth and:

1. for fixed $x, p \rightarrowtail H(x,p)$ is strictly convex function;
2. satisfies the coercivity condition

$$\lim_{|p|\to\infty} \frac{H(x,p)}{|p|} = +\infty$$

and without loss of generality we suppose further that $H(x,p) \geq 1$.

Furthermore, $(u_0, m_0) \in C^\infty(\mathbb{T}^d)$ with $m_0 \geq 0$, and $\fint_{\mathbb{T}^d} m_0 = 1$.

Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ and $\mathbb{R}_0^+ = \{x \in \mathbb{R} | x \geq 0\}$

A2. $g : \mathbb{R}_0^+ \to \mathbb{R}$ is a non-negative increasing function.

From the previous hypothesis it follows that $g(z) = G'(z)$ for some convex increasing function $G : \mathbb{R}_0^+ \to \mathbb{R}$.

We define the Legendre transform of H by

$$L(x,v) = \sup_p(-p \cdot v - H(x,p)). \tag{1.12}$$

Then if we set

$$\hat{L}(x,p) = D_pH(x,p)p - H(x,p), \tag{1.13}$$

by standard properties of the Legendre transform $\hat{L}(x,p) = L(x, -D_pH(x,p))$.

A3. For some $c, C > 0$

$$\hat{L}(x,p) \geq cH(x,p) - C.$$

A4. $g(m) = m^\alpha$, for some $\alpha > 0$.

A5. H satisfies $|D_xH|$, $|D^2_{xx}H| \leq CH + C$, andm for any symmetric matrix M, and any $\delta > 0$ there exists C_δ such that

$$\operatorname{Tr}(D^2_{px}) \leq \delta \operatorname{Tr}(D^2_{pp}HM^2) + C_\delta H.$$

Note that since we assume $H \geq 1$ we can replace the inequality above by $|D_xH|$, $|D^2_{xx}H| \leq \tilde{C}H$, for some constant $\tilde{C}$.

A6. We have $m_0 \geq \kappa_0$, for some $\kappa \in \mathbb{R}^+$.

The next group of hypothesis concerns subquadratic growth.

A7. H satisfies the subquadratic growth condition $H(x,p) = C|p|^\gamma + C$, for some $1 + \frac{1}{d+1} < \gamma < 2$.

A8. D_pH satisfies the subquadratic growth condition $|D_pH| = C|p|^{\gamma-1} + C$, for some $1 + \frac{1}{d+1} < \gamma < 2$.

A9. H satisfies $|D^2_{xp}H|^2 \leq CH$ and, for any symmetric matrix M

$$|D^2_{pp}HM|^2 \leq C\,\mathrm{Tr}(D^2_{pp}HMM).$$

The second assertion in A9 ensures the existence of a uniform upper bound for the eigenvalues of $D^2_{pp}H$.

Observe that, for $d > 2$ and $1 < \gamma < 2$ one has

$$\frac{-4(-4+\gamma)^2(-1+\gamma)\gamma^2 + 2d(-4+(-2+\gamma)\gamma)(-4+(-4+\gamma)(-2+\gamma)\gamma)}{(-2+d)(-4+\gamma)(-1+\gamma)\gamma(-2(-4+\gamma)\gamma + d(-4+(-2+\gamma)\gamma))} > \frac{2}{d-2}$$

A10. The exponent α is such that $0 < \alpha < \alpha_{\gamma,d}$.

2
First and Second Order Estimates

This chapter details two classes of estimates used in the study of time dependent mean-field games. Those are the first and second order estimates.

Let (u^ϵ, m^ϵ) be a solution of (1.7). Then u^ϵ is the value function for the following stochastic optimal control problem

$$u^\epsilon(x,t) = \inf_v E \int_t^T \left[L(x(s), v(s)) + g_\epsilon(m^\epsilon)(x(s), s)\right] ds + u^\epsilon(x(T), T), \quad (2.1)$$

where L is given by (1.12), and the infimum is taken over all bounded and progressively measurable control v,

$$dx = vds + \sqrt{2}dW_s,$$

where $x(t) = x$, and W_s is a d-dimensional Brownian motion. The estimates that we will see over this chapter can be provided as a consequence of this optimal control representation formula.

2.1
Lax-Hopf Estimate

Proposition 2.1.1 *Suppose A1 holds. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then for any smooth vector field $b : \mathbb{T}^d \times (t,T) \to \mathbb{R}^d$, and any solution to*

$$\xi_t + \operatorname{div}(b\xi) = \Delta\xi, \quad (2.2)$$

with $\xi(x,t) = \xi_0$ we have the following upper bound:

$$\begin{aligned} \int_{\mathbb{T}^d} u^\epsilon(x,t)\xi_0 dx \leq \int_t^T \int_{\mathbb{T}^d} (L(y, b(y,s)) + g_\epsilon(m^\epsilon)(y,s))\xi(y,s) dyds \\ + \int_{\mathbb{T}^d} u^\epsilon(y,T)\xi(y,T)dx. \end{aligned} \quad (2.3)$$

Proof. Multiplying the first equation of (1.7) by ξ and 2.2 by u^ϵ, we obtain the auxiliary system:

$$\begin{cases} -u_t^\epsilon \xi + \xi H(x, D_x u^\epsilon) = \xi \Delta u^\epsilon + \xi g_\epsilon(m^\epsilon) \\ \xi_t u^\epsilon + u^\epsilon \operatorname{div}(b\xi) = u^\epsilon \Delta \xi. \end{cases}$$

We subtract these equations to obtain

$$-\xi_t u^\epsilon - \xi u_t^\epsilon = u^\epsilon \operatorname{div}(b\xi) + \xi H(x, D_x u^\epsilon) - u^\epsilon \Delta \xi + \xi \Delta u^\epsilon + \xi g_\epsilon(m^\epsilon). \tag{2.4}$$

Notice that:

$$-\xi_t u^\epsilon - \xi u_t^\epsilon = -\frac{d}{dt}(u^\epsilon \xi);$$

also, integrating (2.4) in $\mathbb{T}^d$ yields

$$\int_{\mathbb{T}^d} u^\epsilon \operatorname{div}(b\xi) dx = \int_{\partial \mathbb{T}^d} u^\epsilon b \xi dx - \int_{\mathbb{T}^d} Du^\epsilon b \xi dx = -\int_{\mathbb{T}^d} Du^\epsilon b \xi dx.$$

Notice we have no boundary terms, because the Torus is a compact manifold without boundary. Moreover,

$$\int_{\mathbb{T}^d} -u^\epsilon \Delta \xi + \int_{\mathbb{T}^d} \xi \Delta u^\epsilon = \int_{\mathbb{T}^d} D\xi D u^\epsilon - \int_{\mathbb{T}^d} D\xi D u^\epsilon = 0.$$

Thus,

$$-\frac{d}{dt} \int_{\mathbb{T}^d} u^\epsilon \xi dx = \int_{\mathbb{T}^d} (-b(x,t) Du^\epsilon - H(x, Du^\epsilon) + g_\epsilon(m^\epsilon)) \xi dx. \tag{2.5}$$

Note that $L(x, v) = \sup(-p \cdot v - H(x, p))$. Because in our problem $v = b$ and $p = Du^\epsilon$, we get $L(x, b) \geq Du^\epsilon \cdot b - H(x, Du^\epsilon)$.

Then,

$$\int_{\mathbb{T}^d} (-b(x,t) Du^\epsilon - H(x, Du^\epsilon) + g_\epsilon(m^\epsilon)) \xi dx \leq \int_{\mathbb{T}^d} (L(x, b) + g_\epsilon(m^\epsilon)) \xi dx \tag{2.6}$$

Now, integrating with respect to the Lebesgue measure in $[t, T]$, we discover

$$\int_t^T -\frac{d}{dt} \int_{\mathbb{T}^d} u^\epsilon \xi dx \leq \int_t^T \int_{\mathbb{T}^d} (L(x, b) + g_\epsilon(m^\epsilon)) \xi dx dt. \tag{2.7}$$

That is,

$$\int_{\mathbb{T}^d} (u^\epsilon(x,t) \xi(x,t) - u^\epsilon(x,T) \xi(x,T)) dx \leq \int_t^T \int_{\mathbb{T}^d} (L(x, b) + g_\epsilon(m^\epsilon)) \xi dx dt$$

We conclude that:

$$\int_{\mathbb{T}^d} u^\epsilon(x,t)\xi_0 dx \leq \int_t^T \int_{\mathbb{T}^d} (L(y, b(y,s)) \\ + g_\epsilon(m^\epsilon)(y,s))\xi(y,s)dyds + \int_{\mathbb{T}^d} u^\epsilon(y,T)\xi(y,T)dx.$$

■

The following corollary shows us what happen when we take $b = 0$, which is a natural choice. Also, we can choose ξ_0 to be either the Lebesgue measure or the measure m_0.

Corollary 2.1.1 *Suppose A1 is in force. Let (u^ϵ, m^ϵ) be a solution to (1.7). Then the following upper bounds are available:*

i) If $\mu(x,t)$ solves the heat equation with $\mu(x,0) = m_0$, we have

$$\int_{\mathbb{T}^d} u^\epsilon(x,0)m_0 dx \leq CT + \int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)(x,t)\mu(x,t)dxdt + \\ \int_{\mathbb{T}^d} u^\epsilon(x,T)\mu(x,T)dx. \tag{2.8}$$

ii) We also have:

$$\int_{\mathbb{T}^d} u^\epsilon(x,0)dx \leq CT + \int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)(x,t)dxdt + \int_{\mathbb{T}^d} u^\epsilon(x,T)dx \tag{2.9}$$

Proof. Choosing $b = 0$, $t = 0$ and $\xi_0 = m_0$ in 2.3, we obtain:

$$\xi_t + \operatorname{div}(b\xi) = \Delta\xi \Rightarrow \xi_t - \Delta\xi = 0,$$

where $\mu(x,t)$ is the solution of the heat equation, and $\mu(x,0) = m_0$.

Even more,

$$L(x,b) = \sup_{p\in\mathbb{R}^d}(p\cdot b - H(x,p));$$

hence $L(x,0) = \sup\limits_{p\in\mathbb{R}^d} -H(x,p) < -1$, as implied by A1.

Thus,

$$\int_0^T \int_{\mathbb{T}^d} L(x,0)dxdt < \int_0^T Cdt = CT$$

and,

$$\int_{\mathbb{T}^d} u^\epsilon(x,0)m_0 dx \leq CT + \int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)\mu(x,t)dxdt + \int_{\mathbb{T}^d} u^\epsilon(x,T)\mu(x,T)dx.$$

Note that, if we take $\xi_0 = 1$ and $b = 0$ on 2.2, we have

$$\int_{\mathbb{T}^d} u^\epsilon(x,0)dx \leq CT + \int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)dxdt + \int_{\mathbb{T}^d} u^\epsilon(x,T)dx,$$

where $\xi = 1$ is the solution of:

$$\begin{cases} \xi_t - \Delta\xi = 0, & \text{in} \quad \mathbb{T}^d \times [0,\infty] \\ \xi(x,0) = 1, & \text{on} \quad \mathbb{T}^d \times \{t = 0\}, \end{cases}$$

which is unique.

■

2.2
First Order Estimates

For a function f we define the oscillation in a given domain $\Omega \in \mathbb{R}^d$ as follows

$$\operatorname{osc}_{x\in\Omega} f(x) = \sup_{x\in\Omega} f(x) - \inf_{x\in\Omega} f(x).$$

Proposition 2.2.1 *Assume A1-3 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then*

$$\int_0^T \int_{\mathbb{T}^d} cH(x, D_x u^\epsilon)m^\epsilon + G(\eta_\epsilon * m_\epsilon)dxdt \leq CT + C \operatorname{osc} u^\epsilon(\cdot, T), \tag{2.10}$$

where $G' = g$.

Proof. Multiplying the first equation of (1.7) by m^ϵ, the second by u^ϵ and subtracting them, we obtain:

$$-(u_t^\epsilon m^\epsilon + m_t^\epsilon u^\epsilon) + m^\epsilon H(x, Du^\epsilon) + u^\epsilon \operatorname{div}(D_p H m^\epsilon) = m^\epsilon \Delta u^\epsilon - u^\epsilon \Delta m^\epsilon + g_\epsilon(m^\epsilon)m^\epsilon.$$

Integrate in the d-dimensional torus $\mathbb{T}^d$ to get

$$\int_{\mathbb{T}^d} -(u_t^\epsilon m^\epsilon + m_t^\epsilon u^\epsilon)dx = -\frac{d}{dt}\int_{\mathbb{T}^d} u^\epsilon m^\epsilon dx,$$

$$\int_{\mathbb{T}^d} u^\epsilon \operatorname{div}(D_p H m^\epsilon) = \int_{\partial\mathbb{T}^d} u^\epsilon D_p H m^\epsilon dx - \int_{\mathbb{T}^d} Du^\epsilon D_p H m^\epsilon dx = -\int_{\mathbb{T}^d} Du^\epsilon D_p H m^\epsilon dx,$$

and

$$\int_{\mathbb{T}^d} m^\epsilon \Delta u^\epsilon - u^\epsilon \Delta m^\epsilon dx = 0.$$

Thus,

$$-\frac{d}{dt}\int_{\mathbb{T}^d} u^\epsilon m^\epsilon dx + \int_{\mathbb{T}^d} (H - Du^\epsilon D_p H)m^\epsilon dx = \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)m^\epsilon dx. \tag{2.11}$$

By A2, we have:

$$\hat{L}(x,p) = D_pH(x,p)p - H(x,p).$$

Hence,

$$-\hat{L}(x, Du^\epsilon) = H(x, Du^\epsilon) - Du^\epsilon D_pH(x, Du^\epsilon).$$

By A3:

$$cH(x, Du^\epsilon) - C \leq Du^\epsilon D_pH(x, Du^\epsilon) - H(x, Du^\epsilon).$$

As consequence,

$$-cH(x, Du^\epsilon) + C \geq H(x, Du^\epsilon) - Du^\epsilon D_pH(x, Du^\epsilon).$$

Taken these observations into account 2.11, becomes:

$$\begin{aligned}\int_{\mathbb{T}^d}(-cH(x,Du^\epsilon)+C)m^\epsilon dx \geq& \int_{\mathbb{T}^d}(H(x,Du^\epsilon) - Du^\epsilon D_pH(x,Du^\epsilon))m^\epsilon dx\\ =& \frac{d}{dt}\int_{\mathbb{T}^d} u^\epsilon m^\epsilon dx + \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)m^\epsilon dx.\end{aligned}$$

Multiplying by (-1) and integrating in $[0,T]$, we get

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}(cH(x,Du^\epsilon)-C)m^\epsilon dxdt \leq& -\int_0^T\int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)m^\epsilon dxdt\\ &+\int_{\mathbb{T}^d}u^\epsilon(x,0)m^\epsilon(x,0)dx - \int_{\mathbb{T}^d}u^\epsilon(x,T)m^\epsilon(x,T)dx.\end{aligned}$$

Note that

$$\int_0^T\int_{\mathbb{T}^d} Cm^\epsilon dxdt = C\int_0^T\int_{\mathbb{T}^d} m^\epsilon dxdt = C\int_0^T 1dt = CT.$$

From Corollary 2.1.1,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} cH(x,Du^\epsilon)m^\epsilon dxdt \leq& CT - \int_0^T\int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)m^\epsilon dxdt + \int_0^T\int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)\mu(x,t)dxdt\\ &+\int_{\mathbb{T}^d}u^\epsilon(x,T)\mu(x,T)dx - \int_{\mathbb{T}^d}u^\epsilon(x,T)m^\epsilon(x,T)dx\\ =& \int_0^T\int_{\mathbb{T}^d} g_\epsilon(m^\epsilon)(\mu - m^\epsilon) + \int_{\mathbb{T}^d}u^\epsilon(x,T)(\mu(x,T)\\ &-m^\epsilon(x,T))dx\end{aligned}$$

Remembering that $g_\epsilon(m^\epsilon) = \eta_\epsilon * g(\eta_\epsilon * m^\epsilon)$, and by A2, there exists a convex function G such that $g(z) = G'(z)$. Also $\eta_\epsilon(y) = \eta_\epsilon(-y)$.

Then,

$$\eta_\epsilon * g(\eta_\epsilon * m^\epsilon)(\mu - m^\epsilon) = g(\eta_\epsilon * m^\epsilon)\eta_\epsilon * (\mu - m^\epsilon) \leq G(\eta_\epsilon * \mu) - G(\eta_\epsilon * m^\epsilon)$$

Hence,

$$\begin{aligned} c\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon)m^\epsilon dxdt \leq CT &+ \int_{\mathbb{T}^d} u^\epsilon(x,T)(\mu(x,T) - m^\epsilon(x,T))dx \\ &+ \int_0^T \int_{\mathbb{T}^d} G(\eta_\epsilon * \mu) - G(\eta_\epsilon * m^\epsilon)dxdt. \end{aligned}$$

Note that

$$\int_{\mathbb{T}^d} u^\epsilon(x,T)\mu(x,T) \leq \sup u^\epsilon \int_{\mathbb{T}^d} \mu(x,T) = \sup u^\epsilon.$$

and,

$$-\int_{\mathbb{T}^d} u^\epsilon(x,T)m^\epsilon(x,T) \geq \inf u^\epsilon \int_{\mathbb{T}^d} m^\epsilon(x,T) = \inf u^\epsilon.$$

We have that μ is bounded, then

$$G(\eta_\epsilon * \mu) = G\left(\int_{\mathbb{T}^d} \eta_\epsilon(y)\mu(x-y)dy\right) \leq G(\|\mu(x-y)\|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} \eta_\epsilon(y)dy).$$

Since η_ϵ is a mollifyer,

$$\int_{\mathbb{T}^d} \eta_\epsilon = 1.$$

Then, $G(\eta_\epsilon * \mu)$ is also bounded, that is,

$$\int_0^T \int_{\mathbb{T}^d} G(\eta_\epsilon * \mu) \leq C.$$

To conclude, we gather the former computations to obtain

$$\begin{aligned} c\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon)m^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^d} G(\eta_\epsilon * m^\epsilon)dxdt &\leq CT + \sup u^\epsilon - \inf u^\epsilon + C \\ &= CT + C \operatorname{osc} u^\epsilon. \end{aligned}$$

■

If we specialize g as $g(z) = z^\alpha$ we obtain further information.

Corollary 2.2.1 *Assume A1-4 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then,*

$$\int_0^T \int_{\mathbb{T}^d} (\eta_\epsilon * m^\epsilon)^{\alpha+1} + H(x, du^\epsilon)m^\epsilon dxdt \leq C. \tag{2.12}$$

Proof. Note that, since $g(m^\epsilon) = (m^\epsilon)^\alpha$, we have

$$G'(\eta_\epsilon * m^\epsilon) = g(\eta_\epsilon * m^\epsilon) = (\eta_\epsilon * m^\epsilon)^\alpha \Rightarrow G(\eta_\epsilon * m^\epsilon) = (\eta_\epsilon * m^\epsilon)^{\alpha+1}.$$

Then,

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) m^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^d} G(\eta_\epsilon * m^\epsilon) dxdt \\ &= \int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) m^\epsilon dxdt + \int_0^T \int_{\mathbb{T}^d} (\eta_\epsilon * m^\epsilon)^{\alpha+1} dxdt \leq C.\end{aligned}$$

Because T is the terminal instant, so is fixed, that is, $CT \leq K_1$, for some constant K_1. Moreover $\operatorname{osc} u^\epsilon(\cdot, T) \leq K_2$, for some constant K_2, because T, is fixed. Then,

$$C = K_1 + K_2.$$

The result follows. ■

2.3 Gains of Regularity for the Hamilton-Jacobi Equation

We will now obtain improved regularity for the Hamilton-Jacobi equation, the first equation of (1.7), by applying the results from the previous section.

Proposition 2.3.1 *Let (u^ϵ, m^ϵ) be a solution to (1.7). Suppose $g \geq 0$ and let $M = \max_x H(x, 0)$. Then,*

$$u^\epsilon(x, t) \geq \min_x u^\epsilon(x, T) + M(t - T). \tag{2.13}$$

Proof. Remind that the first equation of (1.7) is:

$$-u_t^\epsilon + H(x, Du^\epsilon) = \Delta u^\epsilon + g_\epsilon(m^\epsilon).$$

At an extremum point, we have $Du^\epsilon = 0$. Then:

$$\begin{aligned}-u_t^\epsilon - \Delta u^\epsilon = g_\epsilon(m^\epsilon) - H(x, 0) &\geq g_\epsilon(m^\epsilon) - \max_x H(x, 0) \\ &= g_\epsilon(m^\epsilon) + \min_x H(x, 0) \\ &\geq 0.\end{aligned}$$

By the Maximum Principle,

$$u^\epsilon(x,t) \geq \min_x u^\epsilon(x,T),$$

and,

$$-MT \leq -Mt.$$

Add these equations to obtain:

$$u^\epsilon(x,t) - Mt \geq \min_x u^\epsilon(x,T) - MT.$$

Thus,

$$u^\epsilon(x,t) \geq \min_x u^\epsilon(x,T) + M(t-T).$$

■

Remark 1 *For the Maximum Principle, we refer the reader to [2], chapter 2.*

Proposition 2.3.2 *Assume A1-4 are in force. Let (u^ϵ, m^ϵ) be a solution to (1.7). We have:*

$$\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) dxdt \leq C + \int_{T^d} (u^\epsilon(x,T) - u^\epsilon(x,0)) dx. \tag{2.14}$$

Proof. By the Corollary 2.2.1

$$\int_0^T \int_{\mathbb{T}^d} (\eta_\epsilon * m^\epsilon)^{\alpha+1} + H(x, Du^\epsilon) dxdt - C \leq 0.$$

Furthermore, by the Corollary 2.1.1

$$0 \leq \int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon) dxdt + \int_{\mathbb{T}^d} u^\epsilon(x,T) - u^\epsilon(x,0) dx + CT.$$

Thus,

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon) + H(x, Du^\epsilon) dxdt - C \leq \int_0^T \int_{\mathbb{T}^d} g_\epsilon(m^\epsilon) dxdt \\ + \int_{\mathbb{T}^d} u^\epsilon(x,T) - u^\epsilon(x,0) dx + CT.\end{aligned}$$

Hence,

$$\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) dxdt \leq \int_{\mathbb{T}^d} u^\epsilon(x,T) - u^\epsilon(x,0) dx + C.$$

■

Corollary 2.3.1 *Assume A1-4 are in force. Let (u^ϵ, m^ϵ) be a solution to (1.7). Then, the following lower bound is available.*

$$\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) dxdt \leq C + \operatorname{osc} u^\epsilon(\cdot, T) \tag{2.15}$$

and

$$\int_{\mathbb{T}^d} |u^\epsilon(x,0)| dx \leq C + 3 \left\| u^\epsilon(\cdot, T) \right\|_{L^\infty(\mathbb{T}^d)}. \tag{2.16}$$

Proof. By proposition 2.3.1, we have:

$$u^\epsilon(x,t) \geq \min_x u^\epsilon(x,T) + M(t-T) - \max u^\epsilon(x,T).$$

Taken $t = 0$,

$$-u^\epsilon(x,0) \leq \operatorname{osc} u^\epsilon(\cdot, T) + MT.$$

Integrating with respect to the Lebesgue measure to d-dimensional torus $\mathbb{T}^d$, we obtain:

$$\int_{\mathbb{T}^d} -u^\epsilon(x,0) dx \leq \int_{\mathbb{T}^d} (\operatorname{osc} u^\epsilon(\cdot, T) + MT) dx.$$

Taking the modulus, we get

$$\int_{\mathbb{T}^d} |u^\epsilon(x,0)| dx \leq \int_{\mathbb{T}^d} |\operatorname{osc} u^\epsilon(\cdot, T) + MT| dx \leq C + \operatorname{osc} u^\epsilon(\cdot, T),$$

since, $\operatorname{osc} u^\epsilon(\cdot, T) \leq 2 \left\| u^\epsilon(\cdot, T) \right\|_{L^\infty(\mathbb{T}^d)}$.
The result first claim follows.

By 2.14

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) dxdt \leq & C + \int_{T^d} (u^\epsilon(x,T) - u^\epsilon(x,0)) dx \\ \leq & C + \int_{\mathbb{T}^d} (u^\epsilon(x,T) + \operatorname{osc} u^\epsilon(\cdot, T) + MT) dx.\end{aligned}$$

Observe that, if we take $t = T$ in 2.13, we obtain:

$$u^\epsilon(x,T) \geq \min u^\epsilon(x,T).$$

Taken that observation into account, we have:

$$\int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) dxdt \leq C + \operatorname{osc} u^\epsilon(\cdot, T),$$

as we desire. ∎

2.4
Second Order Estimates

In this section we produce second order estimates for a mean-field game system.

Proposition 2.4.1 *Assume A1-6 are in force. Let (u^ϵ, m^ϵ) be a solution to (1.7). Then, the following lower bound is available.*

$$\int_0^T \int_{\mathbb{T}^d} g'(\eta_\epsilon * m^\epsilon)|D_x(\eta_\epsilon * m^\epsilon)|^2 + \mathrm{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)m^\epsilon \le \max_x \Delta u^\epsilon(x,t)$$
$$+ C(1 + \max_x u^\epsilon(x,T) - \min_x u^\epsilon(x,T)) - \int_{\mathbb{T}^d} u^\epsilon(x,0)\Delta m^\epsilon(x,0)dx.$$

Proof. Remember that the first equation of (1.7) is:

$$-u^\epsilon_t + H(x, Du^\epsilon) = \Delta u^\epsilon + g_\epsilon(m^\epsilon).$$

Observe that we need to take the Laplacian Δ in (1.7). Taken the second order derivate of $H(x, D_x u^\epsilon)$, we obtain:

$$\begin{aligned} D_x(D_x(H(x, D_x u^\epsilon))) &= D_x(D_x H + D_p H D^2_{xx} u^\epsilon) \\ &= D^2_{xx}H + D^2_{px}HD^2_{xx}u^\epsilon + (D^2_{xp} + D^2_{pp}D^2_{xx}u^\epsilon)D^2_{xx}u^\epsilon \\ &\quad + D_pHD_xD^2_{xx}u^\epsilon. \end{aligned}$$

Now, apply the Trace operator to conclude

$$\Delta_x H = 2\,\mathrm{Tr}(D^2_{px}HD^2_{xx}u^\epsilon) + \mathrm{Tr}(D^2_{pp}(D^2_{xx}u^\epsilon)^2) + D_pHD_x\Delta u^\epsilon.$$

Note that,

$$g_\epsilon(m^\epsilon) = \eta_\epsilon * g(\eta_\epsilon * m^\epsilon).$$

Then, taken the Laplacian Δ in $g_\epsilon(m^\epsilon)$, we obtain:

$$\Delta g_\epsilon(m^\epsilon) = \mathrm{div}(D_x g_\epsilon(m^\epsilon)) = \mathrm{div}(\eta_\epsilon * (g'(\eta_\epsilon * m^\epsilon) \cdot D_x(\eta_\epsilon * m^\epsilon))).$$

Thus,

$$\begin{aligned} &\Delta u^\epsilon_t - \Delta\Delta u_\epsilon + \mathrm{Tr}(D^2_{pp}D^2_{xx}u^\epsilon)^2) + \Delta_x H + 2\,\mathrm{Tr}(D^2_{px}HD^2_{xx}u^\epsilon) + D_pHD_x\Delta u^\epsilon \\ &= \mathrm{div}(\eta_\epsilon * (g'(\eta_\epsilon * m^\epsilon) \cdot D_x(\eta_\epsilon * m^\epsilon))). \end{aligned}$$

Multiplying by m^ϵ and integrating in the d-dimensional torus $\mathbb{T}^d$.

$$\int_{\mathbb{T}^d} m^\epsilon \Delta u_t^\epsilon - m^\epsilon \Delta\Delta u_\epsilon + \operatorname{Tr}(D^2_{pp} D^2_{xx} u^\epsilon)^2) m^\epsilon + m^\epsilon \Delta_x H$$
$$+ 2\operatorname{Tr}(D^2_{px} H D^2_{xx} u^\epsilon) m^\epsilon + m^\epsilon D_p H D_x \Delta u^\epsilon dx$$
$$= \int_{\mathbb{T}^d} m^\epsilon \operatorname{div}(\eta_\epsilon * (g'(\eta_\epsilon * m^\epsilon) \cdot D_x(\eta_\epsilon * m^\epsilon)))dx.$$

Note that,

$$\begin{aligned}\int_{\mathbb{T}^d} m^\epsilon \operatorname{div}(\eta_\epsilon * (g'(\eta_\epsilon * m^\epsilon) \cdot D_x(\eta_\epsilon * m^\epsilon)))dx &= \int_{\mathbb{T}^d} \eta_\epsilon * (g'(\eta_\epsilon * m^\epsilon) \cdot D_x(\eta_\epsilon * m^\epsilon)) D_x m^\epsilon dx \\ &= \int_{\mathbb{T}^d} (g'(\eta_\epsilon * m^\epsilon) \cdot D_x(\eta_\epsilon * m^\epsilon)) D_x(\eta_\epsilon * m^\epsilon) dx \\ &= \int_{\mathbb{T}^d} g'(\eta_\epsilon * m^\epsilon) \cdot |D_x(\eta_\epsilon * m^\epsilon)|^2 dx.\end{aligned}$$

Moreover,

$$\int_{\mathbb{T}^d} m^\epsilon D_p H D_x \Delta u^\epsilon dx = \int_{\partial\mathbb{T}^d} m^\epsilon D_p H dx - \int_{\mathbb{T}^d} \operatorname{div}(m^\epsilon D_p H) \Delta u^\epsilon dx,$$

Since the d-dimensional torus has no boundary terms, we have the following result:

$$\int_{\mathbb{T}^d} m^\epsilon D_p H D_x \Delta u^\epsilon dx = -\int_{\mathbb{T}^d} \operatorname{div}(m^\epsilon D_p H) \Delta u^\epsilon dx.$$

Observe that,

$$\int_{\mathbb{T}^d} m^\epsilon \Delta\Delta u^\epsilon dx = \int_{\mathbb{T}^d} \Delta m^\epsilon \Delta u^\epsilon dx,$$

and,

$$\int_{\mathbb{T}^d} m^\epsilon \partial_t \Delta u^\epsilon dx = \int_{\mathbb{T}^d} \partial_t (m^\epsilon \Delta u^\epsilon) dx - \int_{\mathbb{T}^d} \partial_t m^\epsilon \Delta u^\epsilon dx$$

Integrating in $[0, T]$, we obtain:

$$\int_0^T \int_{\mathbb{T}^d} m_t^\epsilon \Delta u^\epsilon - \operatorname{div}(m^\epsilon D_p H) \Delta u^\epsilon - \Delta m^\epsilon \Delta u^\epsilon dx dt$$
$$= \int_0^T \int_{\mathbb{T}^d} (m_t^\epsilon - \operatorname{div}(m^\epsilon D_p H) - \Delta m^\epsilon) \Delta u^\epsilon dx dt = 0.$$

Furthermore,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\partial_t(m^\epsilon\Delta u^\epsilon)dxdt &= \int_0^T\frac{d}{dt}\int_{\mathbb{T}^d}m^\epsilon\Delta u^\epsilon dxdt\\ &= \int_{\mathbb{T}^d}m^\epsilon(x,T)\Delta u^\epsilon(x,T)-m^\epsilon(x,0)\Delta u^\epsilon(x,0)dx\\ &= \int_{\mathbb{T}^d}m^\epsilon(x,T)\Delta u^\epsilon(x,T)-u^\epsilon(x,0)\Delta m^\epsilon(x,0)dx.\end{aligned}$$

By A5:

$$2\operatorname{Tr}(D^2_{px}H(D^2_{xx}u^\epsilon)^2)\leq\delta\operatorname{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)+C_\delta H.$$

Finally,

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d}g'(\eta_\epsilon*m^\epsilon)|D_x(\eta_\epsilon*m^\epsilon)|^2+\operatorname{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)m^\epsilon dxdt\\ &\leq\int_0^T\int_{\mathbb{T}^d}(|D^2_{xx}H|+\delta\operatorname{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)+C_\delta H)m^\epsilon dxdt\\ &\quad+\int_{\mathbb{T}^d}m^\epsilon(x,T)\Delta u^\epsilon(x,T)-u^\epsilon(x,0)\Delta m^\epsilon(x,0)dx.\end{aligned}$$

Observe that:

$$\int_{\mathbb{T}^d}m^\epsilon(x,T)\Delta u^\epsilon(x,T)dx\leq\max_x\Delta u^\epsilon(x,T)\int_{\mathbb{T}^d}m^\epsilon(x,T)dx=\max_x\Delta u^\epsilon(x,T).$$

Since

$$\int_{\mathbb{T}^d}m^\epsilon(x,T)dx=1.$$

Choosing $\delta=\frac{1}{2}$ and remembering that by A5, $|D^2_{xx}H|\leq CH+C$, we get

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d}g'(\eta_\epsilon*m^\epsilon)|D_x(\eta_\epsilon*m^\epsilon)|^2+\frac{1}{2}\operatorname{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)m^\epsilon dxdt\\ &\leq C+C\int_0^T\int_{\mathbb{T}^d}Hm^\epsilon dxdt+\max_x\Delta u^\epsilon(x,T)-\int_{\mathbb{T}^d}u^\epsilon(x,0)\Delta m^\epsilon(x,0)dx.\end{aligned}$$

By 2.2.1

$$C\int_0^T\int_{\mathbb{T}^d}Hm^\epsilon dxdt\leq C+C\operatorname{osc}(\cdot,T).$$

Then,

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d}g'(\eta_\epsilon*m^\epsilon)|D_x(\eta_\epsilon*m^\epsilon)|^2+\operatorname{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)m^\epsilon dxdt\\ &\leq C(1+\max_x u^\epsilon(x,T)-\min_x u^\epsilon(x,T))+\max_x\Delta u^\epsilon(x,T)-\int_{\mathbb{T}^d}u^\epsilon(x,0)\Delta m^\epsilon(x,0)dx.\end{aligned}$$

■

The following corollary shows us a computation of Sobolev's Theorem with the result above.

Corollary 2.4.1 *Assume A1-6 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then*

$$\int_0^T \int_{\mathbb{T}^d} g'(\eta_\epsilon * m^\epsilon)|D_x(\eta_\epsilon * m^\epsilon)|^2 dxdt \leq C,$$

and so

$$\int_0^T \|\eta_\epsilon * m^\epsilon\|^{\alpha+1}_{L^{\frac{2^*}{2}(\alpha+1)}(\mathbb{T}^d)} dt \leq C.$$

Proof.

First we can see that

$$\int_{\mathbb{T}^d} u^\epsilon(x,0)\Delta m^\epsilon(x,0)dx \leq \|u^\epsilon(x,0)\|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{T}^d} \Delta m^\epsilon(x,0)dx = 0.$$

And

$$\int_0^T \int_{\mathbb{T}^d} tr(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)m^\epsilon dxdt \leq C,$$

since

$$\int_0^T \int_{\mathbb{T}^d} tr(D^2_{pp}H(D^2_{xx}u^\epsilon)^2)m^\epsilon dxdt \leq tr(D^2_{pp}H(D^2_{xx}u^\epsilon)^2) \int_0^T \int_{\mathbb{T}^d} m^\epsilon dxdt = C.$$

Moreover,

$$C(1 + \max_x u^\epsilon(x,T) - \min_x u^\epsilon(x,T)) + \max_x \Delta u^\epsilon(x,T) = C.$$

Then,

$$\int_0^T \int_{\mathbb{T}^d} g'(\eta_\epsilon * m^\epsilon)|D_x(\eta_\epsilon * m^\epsilon)|^2 dxdt \leq C.$$

Now, we are going to compute the second part of the theorem, but first remember that $g(m) = m^\alpha$. Thus,

$$\int_0^T \int_{\mathbb{T}^d} g'(\eta_\epsilon * m^\epsilon)|D_x(\eta_\epsilon * m^\epsilon)|^2 dxdt = \int_0^T \int_{\mathbb{T}^d} (\eta_\epsilon * m^\epsilon)^{\alpha-1}|D_x(\eta_\epsilon * m^\epsilon)|^2 dxdt =$$
$$\int_0^T \left\|(\eta_\epsilon * m^\epsilon)^{\frac{\alpha-1}{2}} D_x(\eta_\epsilon * m^\epsilon)\right\|_{L^2(\mathbb{T}^d)} dt = \int_0^T \left\|\frac{\alpha+1}{2}(\eta_\epsilon * m^\epsilon)^{\frac{\alpha+1}{2}-\frac{2}{2}} D_x(\eta_\epsilon * m^\epsilon)\right\|_{L^2(\mathbb{T}^d)} dt$$
$$\int_0^T \left\|D_x(\eta_\epsilon * m^\epsilon)^{\frac{\alpha+1}{2}}\right\|_{L^2(\mathbb{T}^d)} dt = \int_0^T \|D_x(\eta_\epsilon * m^\epsilon)\|^{\alpha+1}_{L^{\frac{2}{2}(\alpha+1)}(\mathbb{T}^d)} dt.$$

Finally, by Sobolev's theorem we obtain:

$$\int_0^T ||D_x(\eta_\epsilon * m^\epsilon)||^{\alpha+1}_{L^{\frac{2}{2}(\alpha+1)}(\mathbb{T}^d)} dt \geq \int_0^T ||(\eta_\epsilon * m^\epsilon)||^{\alpha+1}_{L^{\frac{2^*}{2}(\alpha+1)}(\mathbb{T}^d)} dt.$$

■

We included the statement of Sobolev's Theorem in the introduction. See also [2], chapter 5

Corollary 2.4.2 *Assume A1-9 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then*

$$\int_0^T \int_{\mathbb{T}^d} |divD_pH|^2 m^\epsilon dxdt \leq C.$$

Proof. Note that $\operatorname{div}(D_pH) = \operatorname{Tr}(D^2_{pp}HD^2_{xx}u^\epsilon) + \operatorname{Tr}(D^2_{xp}H)$.

Thus,

$$|\operatorname{div}(D_pH)|^2 = 2|\operatorname{Tr}(D^2_{pp}HD^2_{xx}u^\epsilon)|^2 + 2|\operatorname{Tr}(D^2_{xp}H)|^2.$$

Using Assumption 9, we get:

$$2|\operatorname{Tr}(D^2_{pp}HD^2_{xx}u^\epsilon)|^2 + 2|\operatorname{Tr}(D^2_{xp}H)|^2 \leq C\operatorname{Tr}(D^2_{pp}H(D^2_{xx}u^\epsilon)^2) + CH.$$

Now apply Proposition 2.4.1 and Proposition 2.2.1 to complete the proof.

■

3
Regularity for the Fokker-Planck Equation

We begin this section taking note if we integrate the second equation of (1.7) we obtain $\int_{\mathbb{T}^d} m^\epsilon(x,t) = 1$, for all $0 \le t \le T$. Observe that the maximum principle yields that $m^\epsilon \ge 0$ if $m^\epsilon(x,0) \ge 0$.

We will explore in this section various estimates and further integrability for m^ϵ. In Section 3.1 we obtain by the second order estimates, described in the previous section, improved integrability of m^ϵ. In Section 3.2 we obtain L^p norms of D_pH to control the integrability of m^ϵ. These guide us to obtain explicit control for norms of m^ϵ in terms of polynomial expressions in $\|D_pH\|_{L^p(\mathbb{T}^d)}$.

3.1
Regularity by the Second Order Estimates

We begin this with a proposition that will help us through this section:

Proposition 3.1.1 *Assume A1 is in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then*

$$\frac{d}{dt}\int_{\mathbb{T}^d} \varphi(m^\epsilon)dx + \int_{\mathbb{T}^d} \mathrm{div}(D_pH)\varphi^*(m^\epsilon)dx = -\int_{\mathbb{T}^d} \varphi''(m^\epsilon)|D_xm^\epsilon|^2dx.$$

The following theorem will provide us a *priori* estimates for m^ϵ.

Theorem 3.1.2 *Assume A1-9 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then for $d > 2$, $\|m^\epsilon\|_{L^\infty([0,T],L^r(\mathbb{T}^d))}$ is bounded for any $1 \le r < \frac{2^*}{2}$ uniformly in ϵ.*

Proof. In this proof we will omit the ϵ to simplify our notation.

Let β_n be an increasing sequence defined inductively such that $\|m(\cdot,t)\|_{1+\beta_n}$ is bounded. Set $\beta_0 = 0$, so that $\|m(\cdot,t)\|_{1+\beta_0} = 1 \le C$. Let $\beta_{n+1} = \frac{2d}{1+\beta_n}$. Observe that β_n is the n^{th} partial sum of the geometric series with tem $\frac{2^n}{d^n}$. Then $\lim_{n\to\infty}\beta_n = \frac{2}{d-2} = \frac{2^*}{2} - 1$. Since this is a sum of a geometric series with term less than 1.

Set
$$q_n = \frac{2^*}{2}(\beta_{n+1}+1) = \frac{d}{d-2}(\beta_{n+1}+1).$$
Observe that
$$q_n > \frac{d}{d-2}\beta_{n+1} + \beta_{n+1} + 1 > 2\beta_{n+1} + 1.$$
Then we get,

$$\|m\|_{2\beta_{n+1}+1} \leq \|m\|_{1+\beta_n}^{1-\lambda_n} \|m\|_{q_n}^{\lambda_n},$$

for $0 < \lambda_n < 1$, since $\frac{\lambda_n}{q_n} + \frac{1-\lambda_n}{1+\beta_n} = \frac{1}{2\beta_{n+1}+1}$. In particular

$$\lambda_n = \frac{q_n}{q_n - \beta_n - 1}\frac{2\beta_{n+1} - \beta_n}{2\beta_{n+1}+1} = \frac{\beta_{n+1}+1}{2\beta_{n+1}+1}.$$

We have that $\|m\|_{1+\beta_n} \leq C$ then, $\|m\|_{1+\beta_n}^{1-\lambda_n} \leq C$.
Thus,

$$\int_{\mathbb{T}^d} m^{2\beta_{n+1}+1} dx = \|m\|_{2\beta_{n+1}+1}^{2\beta_{n+1}+1} \leq C \|m\|_{q_n}^{\lambda_n(2\beta_{n+1}+1)} = C \|m\|_{q_n}^{\beta_{n+1}+1}. \tag{3.1}$$

Taking $\beta > 0$, using Proposition 3.1.1 with $\varphi(m) = m^{\beta+1}$ we get:

First note that, $\varphi'(m) = (\beta+1)m^{\beta}$ and $\varphi''(m) = \beta(\beta+1)m^{\beta-1}$.

Then,

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta+1}(x,t)dx + \int_{\mathbb{T}^d} \operatorname{div}(D_pH)[(-\beta-1)m^{\beta+1} + m^{\beta+1}]dx = -\int_{\mathbb{T}^d} \beta(\beta+1)m^{\beta+1}|D_x m|^2 dx.$$

Observe that,

$$\int_{\mathbb{T}^d} \operatorname{div}(D_pH)[(-\beta-1)m^{\beta+1} + m^{\beta+1}] = -\beta \int_{\mathbb{T}^d} \operatorname{div}(D_pH) m^{\beta+1} dx$$

and,
$$\beta(\beta+1)m^{\beta+1}|D_x m|^2 = \frac{4\beta}{\beta+1}|D_x m^{\frac{\beta+1}{2}}|^2.$$
Now integrate in $[0,\tau]$ to obtain:

$$\begin{aligned}\int_{T^d} m^{\beta+1}(x,\tau)dx + \frac{4\beta}{\beta+1}\int_0^\tau \int_{\mathbb{T}^d} |D_x m^{\frac{\beta+1}{2}}|^2 dxdt \\ = \int_{\mathbb{T}^d} m^{\beta+1}(x,0)dx + \beta \int_0^\tau \int_{\mathbb{T}^d} \operatorname{div}(D_pH)m^{\beta+1} dxdt.\end{aligned} \tag{3.2}$$

Taking the Young's inequality with δ on $|\operatorname{div}(D_pH)m^{\beta+1}|$ we have

$$\int_{\mathbb{T}^d} |\operatorname{div}(D_pH)m^{\beta+1}|dx = \int_{\mathbb{T}^d} |\operatorname{div}(D_pH)m^{\frac{1}{2}}m^{\beta+\frac{1}{2}}|dx$$
$$\leq C_\delta\left(\int_{\mathbb{T}^d} |\operatorname{div}(D_pH)|^2 m dx\right) + \delta\left(\int_{\mathbb{T}^d} m^{2\beta+1}dx\right), \tag{3.3}$$

where all integrals are evaluated at a fixed time t.

Setting $\beta = \beta_{n+1}$, from (2.1), (2.2) and (2.3) we get for any $\tau \in [0,T]$

$$\int_{T^d} m^{\beta_{n+1}+1}(x,\tau)dx + \frac{4\beta_{n+1}}{\beta_{n+1}+1}\int_0^\tau\int_{\mathbb{T}^d} |D_x m^{\frac{\beta_{n+1}+1}{2}}|^2 dxdt$$
$$= \int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)dx + C_\delta\int_0^\tau\int_{\mathbb{T}^d} |\operatorname{div}(D_pH)|^2 m dxdt + \delta\int_0^\tau \|m\|_{q_n}^{\beta_{n+1}+1}\, dt. \tag{3.4}$$

By Sobolev's theorem we get

$$\|m\|_{q_n}^{\beta_{n+1}+1} = \|m\|_{\frac{2*}{2}(\beta_{n+1}+1)}^{\beta_{n+1}+1} = \left\|m^{\frac{\beta_{n+1}+1}{2}}\right\|_{2*}^2 \leq C\left\|m^{\frac{\beta_{n+1}+1}{2}}\right\|_{W^{1,2}}^2$$
$$= C\left(\left\|m^{\frac{\beta_{n+1}+1}{2}}\right\|_2^2 + \left\|Dm^{\frac{\beta_{n+1}+1}{2}}\right\|_2^2\right)^{\frac{1}{2}.2} = C\left(\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}dx + \int_{\mathbb{T}^d} |Dm^{\frac{\beta_{n+1}+1}{2}}|^2 dx\right) \tag{3.5}$$

From (2.1) and $\int_{\mathbb{T}^d} m(x,t) = 1$, for each fixed time t, and applying Holder's inequality and Young's inequality on $\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}$ we have

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}dx = \int_{\mathbb{T}^d} m^{\beta_{n+1}+\frac{1}{2}}m^{\frac{1}{2}}dx \leq \left(\int_{\mathbb{T}^d} m^{2\beta_{n+1}+1}dx\right)^{\frac{1}{2}}\cdot\left(\int_{\mathbb{T}^d} m^{\frac{1}{2}.2}dx\right)^{\frac{1}{2}}$$
$$\leq C_\zeta + \zeta\int_{\mathbb{T}^d} m^{2\beta_{n+1}+1}dx \leq C_\zeta + \zeta\,\|m\|_{q_n}^{\beta_{n+1}+1}.$$

Thus,

$$\|m\|_{q_n}^{\beta_{n+1}+1} \leq C\int_{\mathbb{T}^d} |Dm^{\frac{\beta_{n+1}+1}{2}}|^2 dx + C_\zeta + \zeta\,\|m\|_{q_n}^{\beta_{n+1}+1}. \tag{3.6}$$

From (2.4) and (2.5), taking δ ζ small enough we have for some $\delta_1 > 0$

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,\tau)dx + \delta_1\int_0^\tau \|m\|_{q_n}^{\beta_{n+1}+1}\, dt$$
$$\leq C + C\int_{\mathbb{T}^d} m^{\beta_{n+1}+1}(x,0)dx + C\int_0^\tau\int_{\mathbb{T}^d} |\operatorname{div}(D_pH)|^2 m dxdt$$

Since by Corollary 2.4.2

$$\int_0^\tau\int_{\mathbb{T}^d} |\operatorname{div}(D_pH)|^2 m dxdt$$

is bounded, the result follows.

■

Corollary 3.1.1 *Assume A1-9 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then, for $-\frac{1}{2} \leq \beta \leq 0$ we have*

$$\int_0^T \int_{\mathbb{T}^d} (m^\epsilon)^{\beta-1} |D_x m^\epsilon|^2 dxdt \leq C. \tag{3.7}$$

Proof. To make the notation less crowded, as before, throughout this proof we will omit the ϵ.

We first note that for $-1 \leq \beta \leq 0$

$$\int_{\mathbb{T}^d} m^{\beta+1} dx \leq C,$$

since for each fixed time t we have that $m(\cdot, t)$ is a probability measure. Then, using identity (2.2), coupled with (2.3) and Corollary 2.4.2 yields

$$\int_0^\tau \int_{\mathbb{T}^d} |D_x m^{\frac{\beta+1}{2}}|^2 dxdt \leq C + C \int_0^\tau \int_{\mathbb{T}^d} m^{2\beta+1} dxdt,$$

and provided $-\frac{1}{2} \leq \beta \leq 0$ the right hand side is bounded. ■

3.2
Regularity by L^p Estimates

In this section we obtain estimates for m^ϵ in $L^\infty([0, T], L^p(\mathbb{T}^d))$ depending polynomially on the L^p-norm of $D_p H$, for $p > \frac{d}{2}$. Because we need explicit estimates, we will prove them in detail. Throughout this Section, we omit the ϵ in proofs for ease of presentation.

Lemma 3.2.1 *Let (u^ϵ, m^ϵ) be a solution of (1.7). Then, for $\beta > 1$, there exist constants c, $C > 0$ such that*

$$\frac{d}{dt} \int_{\mathbb{T}^d} (m^\epsilon)^\beta dx \leq C \int_{\mathbb{T}^d} |D_p H|^2 (m^\epsilon)^\beta dx - c \int_{\mathbb{T}^d} |D_x (m^\epsilon)^{\frac{\beta}{2}}|^2 dx.$$

We now improved integrability of m in terms of the $L^r([0, T], L^p(\mathbb{T}^d))$-norms of $|D_p H|^2$ for $p < \infty$.

Lemma 3.2.2 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume that $\beta \geq \beta_0$ for $\beta_0 > 1$ fixed.*

$$\frac{d}{dt}\int_{\mathbb{T}^d}(m^\epsilon)^\beta dx \leq C\left\||D_pH|^2\right\|_{L^p(\mathbb{T}^d)}\left\|(m^\epsilon)^\beta\right\|_{L^q(\mathbb{T}^d)} - c\int_{\mathbb{T}^d}|D_x(m^\epsilon)^{\frac{\beta}{2}}|^2dx, \quad (3.8)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The result follows by applying Holder inequality on

$$C\int_{\mathbb{T}^d}|D_pH|^2(m^\epsilon)^\beta dx.$$

■

Definition 3.2.1 *Let $1 \leq \beta_0 < \frac{2*}{2} = \frac{d}{d-2}$ be a fixed constant. The sequence $(\beta_n)_{n\in\mathbb{N}}$ is defined inductively by $\beta_{n+1} = \theta\beta_n$, where $\theta > 1$ is a fixed constant.*

Lemma 3.2.3 *Assume that $(\beta_n)_{n\in\mathbb{N}}$ as above and let $1 < q < \frac{d}{d-2}$. Then*

$$\left\|(m^\epsilon)^{\beta_{n+1}}\right\|_q \leq \left(\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_n}dx\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}(m^\epsilon)^{\frac{2^*\beta_{n+1}}{2}}dx\right)^{\frac{2(1-\kappa)}{2^*}},$$

where κ is given by (1.8).

Proof. Holder inequality yields

$$||m^{\beta_{n+1}}||_q = \left(\int_{\mathbb{T}^d}(m)^{\beta_{n+1}q}\right)^{\frac{1}{\beta_{n+1}q}} \leq \left(\int_{\mathbb{T}^d}m^{\beta_n}\right)^{\frac{\kappa}{\beta_n}}\left(\int_{\mathbb{T}^d}m^{\frac{2*\beta_{n+1}}{2}}\right)^{\frac{2(1-\kappa)}{2^*\beta_{n+1}}},$$

where

$$\frac{1}{\beta_{n+1}q} = \frac{\kappa}{\beta_n} + \frac{2(1-\kappa)}{2^*\beta_{n+1}}. \quad (3.9)$$

By rearranging the exponents taking the β_{n+1}'th power

$$\left(\int_{\mathbb{T}^d}m^{\beta_{n+1}q}\right)^q \leq \left(\int_{\mathbb{T}^d}m^{\beta_n}\right)^{\frac{\beta_{n+1}\kappa}{\beta_n}}\left(\int_{\mathbb{T}^d}m^{\frac{2*\beta_{n+1}}{2}}\right)^{\frac{2(1-\kappa)}{2^*}},$$

establishing the result. ■

Lemma 3.2.4 *For any $1 < q < \frac{d}{d-2}$ we have*

$$\left\|(m^\epsilon)^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\kappa)} \leq C\left(\int_{\mathbb{T}^d}\left|D_x\left((m^\epsilon)^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{(1-\kappa)} + C\left\|(m^\epsilon)^{\beta_{n+1}}\right\|_q^{(1-\kappa)}.$$

Proof. From Sobolev's inequality we obtain

$$\left\|(m^\epsilon)^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\kappa)} = \left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx + \int_{\mathbb{T}^d}\left|m^{\frac{\beta_{n+1}}{2}}\right|^2 dx\right)^{2(1-\kappa)\frac{1}{2}}$$
$$\leq 2^p\left(\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx\right)^{(1-\kappa)} + \left(\int_{\mathbb{T}^d} m^{\beta_{n+1}}dx\right)^{(1-\kappa)}\right).$$

Now applying Holder's inequality to last term of the above inequality we obtain

$$\left(\int_{\mathbb{T}^d} m^{\beta_{n+1}}\cdot 1dx\right)^{(1-\kappa)} \leq \left(\int_{\mathbb{T}^d} m^{\beta_{n+1}q}\right)^{\frac{1-\kappa}{q}}\cdot\left(\int_{\mathbb{T}^d} 1^p\right)^{\frac{1-\kappa}{p}} \leq C\left\|m^{\beta_{n+1}}\right\|_q^{(1-\kappa)}.$$

Then the result follows. ∎

Lemma 3.2.5 *Assume that $1 < q < \frac{d}{d-2}$. Then, for any $\delta > 0$ there exists C such that*

$$\left\|(m^\epsilon)^{\beta_{n+1}}\right\|_q \leq C + \delta\left\|(m^\epsilon)^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^2.$$

Proof. Note that, $q < \frac{d}{d-2} = \frac{2^*}{2}$, then $1 < q\beta_{n+1} < \frac{2^*\beta_{n+1}}{2}$.

We can apply Holder's interpolation on $||(m^\epsilon)||_{q\beta_{n+1}}$, to obtain

$$\left(\int_{\mathbb{T}^d} m^{\beta_{n+1}q}dx\right)^{\frac{1}{\beta_{n+1}q}} \leq \left(\int_{\mathbb{T}^d} mdx\right)^\lambda\left(\int_{\mathbb{T}^d} m^{\frac{2^*\beta_{n+1}}{2}}dx\right)^{\frac{2(1-\lambda)}{2^*\beta_{n+1}}},$$

where $0 < \lambda < 1$ solves $\frac{1}{\beta_{n+1}q} = \lambda + \frac{2(1-\lambda)}{2^*\beta_{n+1}}$. Taking the β_{n+1}'th power on above equation, and since m is a probability measure, we get.

$$\left\|m^{\beta_{n+1}}\right\|_q \leq \left\|m^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\lambda)}.$$

Furthermore, noticing that $(1-\lambda) < 1$, Young's inequality with δ yields

$$\left\|m^{\beta_{n+1}}\right\|_q \leq C + \delta\left\|m^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^2,$$

establishing the result. ∎

Proposition 3.2.6 *Assume that $1 < q < \frac{d}{d-2}$*

$$\left\|(m^\epsilon)^{\beta_{n+1}}\right\|_q \leq \left(\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_n}dx\right)^{\theta\kappa}\left[C + C\left(\int_{\mathbb{T}^d}\left|D_x\left((m^\epsilon)^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx\right)^{(1-\kappa)}\right].$$

Proof. By combining both Lemmas 3.2.4 and 3.2.5 and taking the $(1-\kappa)$'th power on the statement of Lemma 3.2.5 we obtain

$$\left\|m^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\kappa)} \leq C + C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx\right)^{1-\kappa} + \delta\left\|m^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\kappa)} \Rightarrow$$
$$(1-\delta)\left\|m^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\kappa)} \leq C + C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx\right)^{(1-\kappa)} \Rightarrow$$
$$\left\|m^{\frac{\beta_{n+1}}{2}}\right\|_{2^*}^{2(1-\kappa)} \leq C + C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx\right)^{(1-\kappa)}$$

Now, multiplying both sides by

$$\left(\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_n} dx\right)^{\theta\kappa},$$

and using Lemma 3.2.3 the result follows.

■

Proposition 3.2.7 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume that $1 < q < \frac{d}{d-2}$. Let $\frac{1}{q}+\frac{1}{p}=1$ and $r\kappa = 1$ where κ is given by 1.8. Then*

$$\frac{d}{dt}\int_{\mathbb{T}^d}(m^\epsilon)^{\beta}_{n+1} dx \leq C + C\left\||D_pH|^2\right\|_{L^p(\mathbb{T}^d)}^r \left(\int_{\mathbb{T}^d}(m^\epsilon)^{\beta_n} dx\right)^{\theta} \tag{3.10}$$

Proof. From the statement of Lemma 3.2.2 and using the last Proposition one obtains that

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,t)dx \leq \left\||D_pH|^2\right\|_p \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)\left[C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{(1-\kappa)} + C\right]$$
$$- c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx.$$

Note that,

$$C\left\||D_pH|^2\right\|_p \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{(1-\kappa)}$$
$$\leq \left[C\left\||D_pH|^2\right\|_p \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{(1-\kappa)}\right]^r$$
$$\leq C\left\||D_pH|^2\right\|_p^r \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta}\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{(r-1)}.$$

Since, $r > 1$ and $r\kappa = 1$. From Corollary 3.1.1 one obtains

$$\left(\int_{\mathbb{T}^d} \left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right) \leq C.$$

Nevertheless, taking those observations into account, we get that

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,t)dx \leq \left\||D_pH|^2\right\|_p \left(\int_{\mathbb{T}^d} m^{\beta_n}\right)\left[C\left(\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2\right)^{(1-\kappa)} + C\right]$$
$$- c\int_{\mathbb{T}^d}\left|D_x\left(m^{\frac{\beta_{n+1}}{2}}\right)\right|^2 dx \leq C\left\||D_pH|^2\right\|_p\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa} + C\left\||D_pH|^2\right\|_p^r\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta}$$
$$C + C\left\||D_pH|^2\right\|_p^r\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta},$$

where the last inequality follow from Young's inequality shown bellow

$$C\left\||D_pH|^2\right\|_p\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\cdot 1 \leq \left[C\left\||D_pH|^2\right\|_p\left(\int_{\mathbb{T}^d} m^{\beta_n}\right)^{\theta\kappa}\right]^r + 1^q$$

■

Proof of Theorem 1.1. The proof follows by induction on n. For $n = 1$ we integrate equation (3.10) with respect to dt over $(0,\tau)$ to obtain

$$\int_{\mathbb{T}^d} m^{\beta_1}(x,\tau)dx \leq C\int_0^\tau \left\||D_pH|^2\right\|_{L^p(\mathbb{T}^d)}^r dt + C \leq C\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^r + C,$$

where we are considering that $\int_{\mathbb{T}^d} m^{\beta_0} \leq C$ for some constant $C > 0$. This verifies our claim for $n = 1$. Then,

$$\frac{d}{dt}\int_{\mathbb{T}^d} m^{\beta_{n+1}}dx \leq C\left\||D_pH|^2\right\|_{L^p(\mathbb{T}^d)}^r\left(C + C\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{r_n}\right)^{\theta}. \quad (3.11)$$

Integrating (3.11) with respect to the Lebesgue measure dt over $(0,\tau)$ one obtains that

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,\tau)dx \leq C\int_0^\tau \left\||D_pH|^2\right\|_{L^p(\mathbb{T}^d)}^r \left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{r_n\theta} dt$$
$$\int_0^\tau \left\||D_pH|^2\right\|_{L^p(\mathbb{T}^d)}^r dt + C.$$

A further application of Holder's inequality leads to

$$\int_{\mathbb{T}^d} m^{\beta_{n+1}}(x,\tau)dx \leq C + C\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{r+r_n\theta},$$

establishing the result.

□

4
Upper Bounds for the Hamilton-Jacobi Equation

In this section we investigate L^∞ bounds for the Hamilton-Jacobi equation. Since by Proposition 2.3.1 any solution of (1.7) is bounded by bellow, to get the bounds it is enough to establish upper bounds. These build upon the improved integrability obtain previously for m^ϵ and will be used in the following sections. As before, we omit the ϵ in the proofs in this Chapter.

Proposition 4.0.1 *Suppose (u^ϵ, m^ϵ) is a solution of (1.7) and H satisfies A1. Then, if $p > \frac{d}{2}$, we have*

$$u^\epsilon(x,\tau) \leq (T-\tau)\max_z L(z,0) + C\,||g_\epsilon(m)||_{L^\infty([0,T];L^p(\mathbb{T}^d))} + \int_{\mathbb{T}^d} u^\epsilon(y,T)\theta(y,T-\tau)dy,$$

where θ is the heat kernel, with $\theta(\cdot,\tau) = \delta_x$. Furthermore, if $\frac{1}{r}+\frac{1}{s} = \frac{1}{p}+\frac{1}{q} = 1$, and $\frac{p}{s} > \frac{d}{2}$, we have

$$u^\epsilon(x,\tau) \leq (T-\tau)\max_z L(z,0) + C\,||g_\epsilon(m)||_{L^r([0,T];L^p(\mathbb{T}^d))} + \int_{\mathbb{T}^d} u^\epsilon(y,T)\theta(y,T-\tau)dy.$$

Proof. By applying Proposition 2.1.1 with $b = 0$ and $\zeta_0 = \theta(\cdot,\tau) = \delta_x$, we obtain the estimate

$$\begin{aligned} u(x,\tau) \leq{} & (T-\tau)\max_{z\in\mathbb{T}^d} L(z,0) \\ & + \int_\tau^T \int_{\mathbb{T}^d} g(m)(y,t)\theta(y,t-\tau)dydt + \int_{\mathbb{T}^d} u(y,T)\theta(y,T-\tau)dy. \end{aligned}$$

where,

$$\begin{aligned} \int_\tau^T \int_{\mathbb{T}^d} L(y,0)\theta(y,t-\tau)dydt &\leq \max_{z\in\mathbb{T}^d} L(z,0)\int_\tau^T \int_{\mathbb{T}^d} \theta(y,t-\tau)dydt = \max_{z\in\mathbb{T}^d} L(z,0)\int_\tau^T 1dt \\ &= (T-\tau)\max_{z\in\mathbb{T}^d} L(z,0). \end{aligned}$$

Now, we need to estimate $\int_\tau^T \int_{\mathbb{T}^d} g(m)(y,t)\theta(y,t-\tau)dydt$. To doing so, we recall the following property of the heat kernel, for $\frac{1}{p}+\frac{1}{q} = 1$ we have $||\theta(\cdot,t)||_q \leq \frac{C}{t^{\frac{d}{2p}}}$.

Hence, Holder's inequality yields

$$\int_{\mathbb{T}^d} g(m)(y,t)\theta(y,t-\tau)dy \leq \|g(m(\cdot,t))\|_{L^p(\mathbb{T}^d)} \|\theta(\cdot,t)\|_{L^q(\mathbb{T}^d)} \leq \frac{C}{(t-\tau)^{\frac{d}{2p}}} \|g(m(\cdot,t))\|_{L^p(\mathbb{T}^d)}.$$

Thus, if $d < 2p$ we have

$$\int_\tau^T \int_{\mathbb{T}^d} g(m)(y,t)\theta(y,t-\tau)dydt \leq \int_\tau^T \frac{C}{(t-\tau)^{\frac{d}{2p}}} \|g(m(\cdot,t))\|_{L^p(\mathbb{T}^d)} dt \leq C \|g(m)\|_{L^\infty([0,T];L^p(\mathbb{T}^d))}.$$

For the second assertion, Holder's inequality leads to

$$\int_\tau^T \int_{\mathbb{T}^d} g(m)(y,t)\theta(y,t-\tau)dydt \leq \int_\tau^T \|g(m)(\cdot,t)\|_{L^p(\mathbb{T}^d)} \|\theta(\cdot,t-\tau)\|_{L^q(\mathbb{T}^d)} dt$$
$$\leq \|g(m)\|_{L^r([0,T];L^p(\mathbb{T}^d))} \left(\int_\tau^T \frac{C}{t^{\frac{ds}{2p}}}\right)^{\frac{1}{s}} \leq C \|g(m)\|_{L^r([0,T];L^p(\mathbb{T}^d))},$$

where the last inequality follows from $\frac{ds}{2p} < 1$. ■

Corollary 4.0.1 *Suppose A1-6 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7). Then, for any $p > \frac{d}{2}$ we have*

$$u(x,\tau) \leq (T-\tau)\max_{z\in\mathbb{T}^d} L(z,0) + C\|\eta_\epsilon * m^\epsilon\|^\alpha_{L^\infty([0,T];L^{\alpha p}(\mathbb{T}^d))} \int_{\mathbb{T}^d} u(y,T)\theta(y,T-\tau)dy,$$

where θ is the heat kernel, with $\theta(\cdot,\tau) = \delta_x$. Furthermore, if $\alpha p \leq 1$ we have

$$u(x,\tau) \leq (T-\tau)\max_{z\in\mathbb{T}^d} L(z,0) + C + \int_{\mathbb{T}^d} u(y,T)\theta(y,T-\tau)dy.$$

Proof. To begin our proof, first note that by A4, $g(m) = m^\alpha$.

Thus,

$$\|g_\epsilon(m)\|_p = \|\eta_\epsilon * g(\eta_\epsilon * m)\|_p = ||(\eta_\epsilon * m)^\alpha||_p = \left(\int_{\mathbb{T}^d} |\eta_\epsilon * m|^{\alpha p} dx\right)^{\frac{1}{p}=\frac{\alpha}{\alpha p}} = \|\eta_\epsilon * m\|^\alpha_{\alpha p}.$$

Then the first assertion follows.

Nevertheless, to establish the second assertion, note that, since $\alpha p \leq 1$

$$\|\eta_\epsilon * m\|^\alpha_{\alpha p} \leq \|\eta_\epsilon * m\|^\alpha_1 = \left(\int_{\mathbb{T}^d} (\eta_\epsilon * m)dx\right)^\alpha = \left(\int_{\mathbb{T}^d} \eta_\epsilon dx\right)^\alpha \left(\int_{\mathbb{T}^d} m dx\right)^\alpha = 1.$$

However, with the observation above, we conclude our proof. ■

Corollary 4.0.2 *Suppose A1-6 are in force. Let (u^ϵ, m^ϵ) be a solution of (1.7).*

Then, for any p, r *such that,* $\frac{p(r-1)}{r} > \frac{d}{2}$*, we have*

$$u(x,\tau) \leq (T-\tau)\max_{z\in\mathbb{T}^d} L(z,0) + C\,\|\eta_\epsilon * m^\epsilon\|^{\alpha}_{L^{\alpha r}([0,T];L^{\alpha p}(\mathbb{T}^d))}\int_{\mathbb{T}^d} u(y,T)\theta(y,T-\tau)dy,$$

Proof. By the second assertion of Proposition 4.0.1 and A4, we have that

$$||g_\epsilon(m)||_{L^r([0,T];L^p(\mathbb{T}^d))} = \|(\eta_\epsilon * m)^\alpha\|_{L^r([0,T];L^p(\mathbb{T}^d))} = \left(\int_0^T \left(\left(\int_{\mathbb{T}^d}(\eta_\epsilon * m)^{\alpha p}dx\right)^{\frac{1}{p}}\right)^r dt\right)^{\frac{1}{r}}$$

$$= \left(\int_0^T \left(\int_{\mathbb{T}^d}(\eta_\epsilon * m)^{\alpha p}dx\right)^{\frac{r}{p}=\frac{\alpha r}{\alpha p}} dt\right)^{\frac{1}{r}} = \left(\int_0^T \|\eta_\epsilon * m\|^{\alpha r}_{L^{\alpha p}(\mathbb{T}^d)}\,dt\right)^{\frac{1}{r}=\frac{\alpha}{\alpha r}} = \|\eta_\epsilon * m\|^{\alpha}_{L^{\alpha r}([0,T];L^{\alpha p}(\mathbb{T}^d))}$$

Then, the result follows. ■

To finish this Chapter we show the proof of Lemma 1.3.1.

Proof of Lemma 1.3.1:

It follows from the second assertion of Proposition 4.0.1. Since $\frac{b(a-1)}{a} > \frac{d}{2}$ holds.

□

5
Sobolev Regularity for the Hamilton-Jacobi Equation

In this Chapter we consider regularity in Sobolev spaces for the Hamilton-Jacobi equation. As before, we omit the ϵ in the proofs in this Chapter.

Lemma 5.0.1 *Let $u \in W^{2,p}(\mathbb{T}^d)$. Then, there exists $C > 0$ such that*

$$\|Du\|_{L^{\gamma p}(\mathbb{T}^d)} \leq C \left\|D^2 u\right\|_{L^p(\mathbb{T}^d)}^{\frac{1}{2}} \|u\|_{L^\infty(\mathbb{T}^d)}^{\frac{1}{2}} . \tag{5.1}$$

Proof. Note that Gagliardo-Nirenberg interpolation Theorem leads to

$$\|D^j u\|_{L^p} \leq C\|D^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha},$$

where,

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q}.$$

Taking $q = \infty$, $j = 1$ and $\alpha = \frac{1}{2}$, the result follow. ■

Lemma 5.0.2 *Let (u^ϵ, m^ϵ) be a solution of (1.7). Then*

$$\|u_t^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))}, \|D^2 u^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq \|g_\epsilon(m^\epsilon)\|_{L^r([0,T];L^p(\mathbb{T}^d))} + \|H\|_{L^r([0,T];L^p(\mathbb{T}^d))},$$

for $1 < p, r < \infty$. Furthermore,

$$\|D^2 u^\epsilon\|_{L^\infty([0,T];L^2(\mathbb{T}^d))} \leq \|g_\epsilon(m^\epsilon)\|_{L^2([0,T];L^2(\mathbb{T}^d))} + \|H\|_{L^2([0,T];L^2(\mathbb{T}^d))}.$$

Lemma 5.0.3 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume that A1-9 are in force. For $1 < p, r < \infty$ there are constants $c, C > 0$ such that*

$$\|H(x, Du^\epsilon)\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq c\|D^2 u^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{\frac{\gamma}{2}} \|u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{T}^d))}^{\frac{\gamma}{2}}.$$

Proof. Assumption A7 yields

$$\left(\int_{\mathbb{T}^d} |H(x, Du(x,t))|^p dx\right)^{\frac{1}{p}} \leq c\left(\int_{\mathbb{T}^d} |Du|^{\gamma p} dx\right)^{\frac{1}{p}} + C.$$

By combining this with Lemma 5.0.1 if follows that

$$\left(\int_{\mathbb{T}^d} |H(x, Du(x,t))|^p dx\right)^{\frac{1}{p}} \leq c\left(\int_{\mathbb{T}^d} |Du|^{\gamma p} dx\right)^{\frac{1}{p}=\frac{\gamma}{\gamma p}} + C = c\|Du\|^{\gamma}_{L^{\gamma p}(\mathbb{T}^d)} + C$$
$$\leq c\|D^2u\|^{\frac{\gamma}{2}}_{L^p(\mathbb{T}^d)}\|u\|^{\frac{\gamma}{2}}_{L^\infty(\mathbb{T}^d)} + C.$$

Thus,

$$\|H(x,Du)\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq c\left(\int_0^T \left(\|D^2u\|^{\frac{\gamma}{2}}_{L^p(\mathbb{T}^d)}\|u\|^{\frac{\gamma}{2}}_{L^\infty(\mathbb{T}^d)}\right)^r dt\right)^{\frac{1}{r}} + C$$
$$\leq c\left(\|u\|^{\frac{\gamma}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} \cdot \int_0^T \left(\|D^2u\|^{\frac{\gamma}{2}}_{L^p(\mathbb{T}^d)}\right)^r dt\right)^{\frac{1}{r}} + C$$
$$\leq c\|D^2u^\epsilon\|^{\frac{\gamma}{2}}_{L^r([0,T];L^p(\mathbb{T}^d))}\|u^\epsilon\|^{\frac{\gamma}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + C,$$

where in the last inequality we used that $\frac{\gamma}{2} < 1$. ■

Proof of Theorem 1.3.2. By combining Lemma 5.0.2 and 5.0.3 yield

$$\|D^2u\|^{\frac{\gamma}{2}}_{L^r([0,T];L^p(\mathbb{T}^d))} \leq c\|D^2u\|^{\frac{\gamma}{2}}_{L^r([0,T];L^p(\mathbb{T}^d))}\|u\|^{\frac{\gamma}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + \|g(m)\|_{L^r([0,T];L^p(\mathbb{T}^d))} + C.$$

Set $j = \frac{2}{\gamma}$ and define l by $\frac{1}{j} + \frac{1}{l} = 1$. Using Young's inequality with δ.

But first, observe that, since $j = \frac{2}{\gamma}$. Then $l = \frac{2}{2-\gamma}$.

Thus,

$$\|D^2u\|^{\frac{\gamma}{2}}_{L^r([0,T];L^p(\mathbb{T}^d))} \leq \delta\|D^2u\|_{L^r([0,T];L^p(\mathbb{T}^d))} + C\|u\|^{\frac{\gamma}{2-\gamma}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + \|g(m)\|_{L^r([0,T];L^p(\mathbb{T}^d))} + C.$$

Absorbing the term $\delta\|D^2u\|_{L^r([0,T];L^p(\mathbb{T}^d))}$ on the left-hand side yields

$$\|D^2u\|^{\frac{\gamma}{2}}_{L^r([0,T];L^p(\mathbb{T}^d))} \leq +c\|u\|^{\frac{\gamma}{2-\gamma}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + c\|g(m)\|_{L^r([0,T];L^p(\mathbb{T}^d))} + C,$$

which concludes the proof.

□

6
Improved Regularity

Throughout this chapter we define, for $1 \leq \beta_0 < \frac{2^*}{2}$ and

$$0 \leq \upsilon \leq 1 < \theta, \tag{6.1}$$

$$a_\upsilon \doteq \frac{\alpha+1}{1-\upsilon} \quad \text{and} \quad b_\upsilon \doteq \frac{d(\alpha+1)\beta_0\theta}{(\alpha+1)d\upsilon + \theta\beta_0(d-2)(d-\upsilon)}. \tag{6.2}$$

As before, we omit the ϵ in the proofs in this Chapter.

Lemma 6.0.1 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Suppose further that a_υ and b_υ are given as in 6.2. Then,*

$$\|m^\epsilon\|_{L^{a_\upsilon}([0,T];L^{b_\upsilon}(\mathbb{T}^d))} \leq C + C \left\| |D_pH|^2 \right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{\frac{r\upsilon(1-\frac{1}{\theta})}{\beta_0(\theta-1)}},$$

where

$$p > \frac{d}{2} \quad \textit{and} \quad r = \frac{p(d(\theta-1)+2)}{2p-d}. \tag{6.3}$$

Proof. Since , $0 \leq \upsilon \leq 1$, $\frac{1}{a_\upsilon} = \frac{1-\upsilon}{\alpha+1}$ and $\frac{1}{b_\upsilon} = \frac{1-\upsilon}{\frac{2^*(\alpha+1)}{2}} + \frac{\upsilon}{\theta\beta_0}$, which hold by 6.2. Holder's inequality implies that,

$$\begin{aligned}
\|m^\epsilon\|_{L^{a_\upsilon}([0,T];L^{b_\upsilon}(\mathbb{T}^d))} &= \left(\int_0^T \left(\left(\int_{\mathbb{T}^d} m^{b_\upsilon} dx\right)^{\frac{1}{b_\upsilon}}\right)^{a_\upsilon} dt\right)^{\frac{1}{a_\upsilon}} \\
&\leq \left(\int_0^T \left(\left(\int_{\mathbb{T}^d} m^{\frac{2^*(\alpha+1)}{2}} dx\right)^{\frac{2}{2^*(\alpha+1)}.(1-\upsilon)} \cdot \left(\int_{\mathbb{T}^d} m^{\theta\beta_0} dx\right)^{\frac{1}{\theta\beta_0}.\upsilon}\right)^{a_\upsilon} dt\right)^{\frac{1}{a_\upsilon}} \\
&= \left(\int_0^T \left(\int_{\mathbb{T}^d} m^{\frac{2^*(\alpha+1)}{2}} dx\right)^{\frac{2}{2^*}} \cdot \left(\int_{\mathbb{T}^d} m^{\theta\beta_0} dx\right)^{\frac{1}{\theta\beta_0}.\upsilon.a_\upsilon} dt\right)^{\frac{1}{a_\upsilon}} \\
&= \left(\int_0^T \|m\|_{L^{\frac{2^*(\alpha+1)}{2}}}^{(\alpha+1)} \cdot \|m\|_{L^{\theta\beta_0}}^{\upsilon.a_\upsilon} dt\right)^{\frac{1}{a_\upsilon}}.
\end{aligned}$$

Now, applying Holder's inequality, with $p = 1$ and $q = \infty$ one obtains that

$$\left(\int_0^T \|m\|^{(\alpha+1)}_{L^{\frac{2^*(\alpha+1)}{2}}} \cdot \|m\|^{v.a_v}_{L^{\theta\beta_0}} dt\right)^{\frac{1}{a_v}} \leq \left(\|m\|^{v.a_v}_{L^\infty([0,T];L^{\theta\beta_0})} \cdot \int_0^T \|m\|^{(\alpha+1)=\frac{(\alpha+1)(1-v)}{1-v}=a_v(1-v)}_{L^{\frac{2^*(\alpha+1)}{2}}} dt\right)^{\frac{1}{a_v}}$$
$$= \|m^{(1-v)}\|_{L^{a_v}([0,T];L^{\frac{2^*(\alpha+1)}{2}}(\mathbb{T}^d))} \cdot \|m\|^v_{L^\infty([0,T];L^{\theta\beta_0}(\mathbb{T}^d))}.$$

Note that, $(1-v)a_v = (\alpha+1)$. Thus,

$$\|m^{(1-v)}\|_{L^{a_v}} = \left(\int m^{(1-v)a_v}\right)^{\frac{1}{a_v}=\frac{1}{a_v}\frac{(1-v)}{(1-v)}} = \left(\int m^{(\alpha+1)}\right)^{(\alpha+1)(1-v)} = \|m\|^{(1-v)}_{L^{(\alpha+1)}}.$$

Nevertheless,

$$\|m^\epsilon\|_{L^{a_v}([0,T];L^{b_v}(\mathbb{T}^d))} \leq \|m\|^{(1-v)}_{L^{(\alpha+1)}([0,T];L^{\frac{2^*(\alpha+1)}{2}}(\mathbb{T}^d))} \cdot \|m\|^v_{L^\infty([0,T];L^{\theta\beta_0}(\mathbb{T}^d))}.$$

Because of Corollary 2.4.1 we have $\|m\|^{(1-v)}_{L^{(\alpha+1)}([0,T];L^{\frac{2^*(\alpha+1)}{2}}(\mathbb{T}^d))} \leq C$. On the other hand, from Theorem 1.1 it follows that

$$\|m\|^v_{L^\infty([0,T];L^{\theta\beta_0}(\mathbb{T}^d))} \leq C + C\left\||D_pH|^2\right\|^{\frac{rv(1-\frac{1}{\theta})}{\beta_0(\theta-1)}}_{L^r([0,T];L^p(\mathbb{T}^d))}.$$

By combining the previous computations one obtains the result. ∎

The following Lemma shows us upper bound for u^ϵ depending on $g_\epsilon(m)$

Lemma 6.0.2 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Assume that*

$$\frac{b_v}{a_v}\left(\frac{a_v-\alpha}{\alpha}\right) > \frac{d}{2}. \tag{6.4}$$

Then,

$$\|u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} \leq C + C\|g_\epsilon(m)\|_{L^{\frac{a_v}{\alpha}}([0,T];L^{\frac{b_v}{\alpha}}(\mathbb{T}^d))}.$$

Proof. The result easily follows from Lemma 1.3.1 since (6.4) holds. ∎

The Lemma below, we can derive more details about upper bounds of u^ϵ and $g_\epsilon(m)$.

Lemma 6.0.3 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Let ζ, $\tilde{p}$ and $\tilde{r}$ such that*

$$0 \leq \zeta \leq 1, \quad \tilde{p}\left(\frac{\tilde{r}-1}{\tilde{r}}\right) > \frac{d}{2}, \tag{6.5}$$

where

$$\frac{1}{\tilde{p}} \doteq \frac{1-\zeta}{\left(1+\frac{1}{\alpha}\right)\frac{d}{d-2}} + \frac{\zeta}{\frac{b_{\upsilon}}{\alpha}}, \tag{6.6}$$

and

$$\frac{1}{\tilde{r}} \doteq \frac{1-\zeta}{1+\frac{1}{\alpha}} + \frac{\zeta}{\frac{a_{\upsilon}}{\alpha}}. \tag{6.7}$$

Then

$$\|g_\epsilon\|_{L^{\tilde{r}}([0,T];L^{\tilde{p}}(\mathbb{T}^d))} \leq C\|g_\epsilon(m)\|^{\zeta}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))},$$

and

$$\|u^\epsilon\|_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} \leq C + C\,\|g_\epsilon\|_{L^{\tilde{r}}([0,T];L^{\tilde{p}}(\mathbb{T}^d))}\,.$$

Proof. Note that the second assertion follows from (6.5) along with Lemma **??**. To solve the first assertion, we can use Holder's inequality

$$\|g_\epsilon\|_{L^{\tilde{r}}([0,T];L^{\tilde{p}}(\mathbb{T}^d))} \leq \|g_\epsilon\|^{(1-\zeta)}_{L^{1+\frac{1}{\alpha}}\left([0,T];L^{\frac{2^*}{2}\left(1+\frac{1}{\alpha}\right)}(\mathbb{T}^d)\right)}\|g_\epsilon(m)\|^{\zeta}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))}.$$

Also, we have from Corollary 2.4.1 that $\|g_\epsilon\|^{(1-\zeta)}_{L^{1+\frac{1}{\alpha}}\left([0,T];L^{\frac{2^*}{2}\left(1+\frac{1}{\alpha}\right)}(\mathbb{T}^d)\right)} < C$, for some $C > 0$. Then combining these, the result follows. ■

The next Lemma we derive upper bound for $|D_pH|^2$.

Lemma 6.0.4 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Suppose further that $p > \frac{d}{2}$ and r is given as in (6.3). Then*

$$\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq C + C\,\|Du^\epsilon\|^{2(1-\lambda)(\gamma-1)}_{L^F([0,T];L^G(\mathbb{T}^d))}\,,$$

where

$$0 \leq \lambda \leq 1, \qquad \frac{1}{2(\gamma-1)r} = \frac{\lambda}{\gamma} + \frac{1-\lambda}{F} \tag{6.8}$$

and

$$\frac{1}{2(\gamma-1)p} = \frac{\lambda}{\gamma} + \frac{1-\lambda}{G}, \tag{6.9}$$

respectively.

Proof. Note that A8 yields,

$$\left\| |D_pH|^2 \right\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq C + C \left\| Du^\epsilon \right\|^{2(1-\lambda)(\gamma-1)}_{L^{2(\gamma-1)r}([0,T];L^{2(\gamma-1)p}(\mathbb{T}^d))}.$$

On the other hand, Holder's inequality implies that

$$\|Du^\epsilon\|^{2(1-\lambda)(\gamma-1)}_{L^{2(\gamma-1)r}([0,T];L^{2(\gamma-1)p}(\mathbb{T}^d))} \leq \|Du^\epsilon\|^{\lambda}_{L^\gamma([0,T];L^\gamma(\mathbb{T}^d))} \|Du^\epsilon\|^{(1-\lambda)}_{L^F([0,T];L^G(\mathbb{T}^d))} \tag{6.10}$$

since (6.8) and (6.9) hold. We have from Proposition 2.2.1 that $Du \in L^\gamma(\mathbb{T}^d \times [0,T])$. Combining these with the computation above, we get the result. ■

Finally, in the next Lemma, we are able to derive upper bound for Du^ϵ.

Lemma 6.0.5 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Suppose further that (6.4) - (6.9),*

$$\frac{F}{\gamma} = \frac{a_\upsilon}{\alpha} \tag{6.11}$$

and

$$\frac{G}{\gamma} = \frac{b_\upsilon}{\alpha} \tag{6.12}$$

hold. Then,

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C\|g_\epsilon\|^{\frac{\zeta}{2-\gamma}+\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + C\|g_\epsilon\|^{\frac{1}{\gamma}+\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + C.$$

Proof. Inequality (5.1) implies that

$$||Du||_{L^{2(\gamma-1)p}(\mathbb{T}^d)} \leq C||D^2u||^{\frac{1}{2}}_{L^{\frac{2(\gamma-1)}{\gamma}}(\mathbb{T}^d)} ||u||^{\frac{1}{2}}_{L^\infty(\mathbb{T}^d)}.$$

Note that $\gamma < 2$ it follows that

$$||Du||_{L^{2(\gamma-1)p}(\mathbb{T}^d)} \leq C||D^2u||^{\frac{1}{\gamma}}_{L^{\frac{2(\gamma-1)}{\gamma}}(\mathbb{T}^d)} ||u||^{\frac{1}{2}}_{L^\infty(\mathbb{T}^d)} + C||u||^{\frac{1}{2}}_{L^\infty(\mathbb{T}^d)}. \tag{6.13}$$

From (6.13) it follows that

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C\|D^2u\|^{\frac{1}{\gamma}}_{L^{\frac{F}{\gamma}}([0,T];L^{\frac{G}{\gamma}}(\mathbb{T}^d))} \|u\|^{\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + C\|u\|^{\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))}$$

and

$$\|D^2u\|^{\frac{1}{\gamma}}_{L^{\frac{F}{\gamma}}([0,T];L^{\frac{G}{\gamma}}(\mathbb{T}^d))} \leq C\|g_\epsilon\|^{\frac{1}{\gamma}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + \|u\|^{\frac{1}{2-\gamma}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + C.$$

By combining these, one obtains

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C\|g_\epsilon\|^{\frac{1}{\gamma}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))}\|u\|^{\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} \tag{6.14}$$

$$+ \|u\|^{\frac{1}{2-\gamma}+\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} + C\|u\|^{\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))}. \tag{6.15}$$

Because of Lemma (6.0.3) we also have

$$\|u\|^{\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))} \leq C + C\|g_\epsilon\|^{\zeta}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))}.$$

Hence, (6.14) becomes

$$\begin{aligned}\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq\ & C\|g_\epsilon\|^{\frac{1}{\gamma}+\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} \\ & + C\|g_\epsilon\|^{\frac{\zeta}{2-\gamma}+\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + \|g_\epsilon\|^{\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} \\ & + C\|g_\epsilon\|^{\frac{1}{\gamma}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + C \\ \leq\ & C\|g_\epsilon\|^{\frac{\zeta}{2-\gamma}+\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + C\|g_\epsilon\|^{\frac{1}{\gamma}+\frac{\zeta}{2}}_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} + C,\end{aligned}$$

where the last inequality follows from Young's inequality applied to those terms with lower exponents. ■

In the next two corollaries we can see more details about upper bounds of $g_\epsilon(m)$ and Du^ϵ.

Corollary 6.0.1 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Suppose further that (6.4) holds. Then,*

$$\|g_\epsilon(m)\|_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} \leq C + C\left\||D_pH|^2\right\|^{\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}}_{L^r([0,T];L^p(\mathbb{T}^d))},$$

where $p > \frac{d}{2}$ and r is given by (6.3)

Proof. Lemma 6.0.1 along with A4 leads to

$$\|g_\epsilon(m)\|_{L^{\frac{a_\upsilon}{\alpha}}([0,T];L^{\frac{b_\upsilon}{\alpha}}(\mathbb{T}^d))} \leq \|m\|^{\alpha}_{L^{a_\upsilon}([0,T];L^{b_\upsilon}(\mathbb{T}^d))} \leq C + C\left\||D_pH|^2\right\|^{\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}}_{L^r([0,T];L^p(\mathbb{T}^d))}$$

and then the result is established. ■

Corollary 6.0.2 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-9 are in force. Suppose further that (6.4) - (6.12) hold. Then,*

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C + C\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))}^{\frac{(1-\lambda)(\gamma-1)(4\zeta-\gamma\zeta)}{(2-\gamma)}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}} + C\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))}^{\frac{(1-\lambda)(\gamma-1)(2+\gamma\zeta)}{\gamma}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}},$$

where $p > \frac{d}{2}$ and r is given by (6.3).

Proof. Lemma 6.0.5 along with Corollary 6.0.1 leads to

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C + C\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{\frac{(4\zeta-\gamma\zeta)}{2(2-\gamma)}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}} + C\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{\frac{(2+\gamma\zeta)}{2\gamma}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}}.$$

Furthermore, because of A8 and Lemma 6.0.4

$$\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{\frac{(4\zeta-\gamma\zeta)}{2(2-\gamma)}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}} \leq C + C\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))}^{\frac{(1-\lambda)(\gamma-1)(4\zeta-\gamma\zeta)}{(2-\gamma)}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}}$$

and

$$\left\||D_pH|^2\right\|_{L^r([0,T];L^p(\mathbb{T}^d))}^{\frac{(2+\gamma\zeta)}{2\gamma}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}} \leq C + C\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))}^{\frac{(1-\lambda)(\gamma-1)(2+\gamma\zeta)}{\gamma}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}}.$$

The result follows by combining both above computation. ∎

This final Lemma below shows us that $\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C$.

Lemma 6.0.6 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-10 are in force. Then,*

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C,$$

where F and G are given by (6.8) and (6.9), respectively.

Proof. By Corollary 6.0.2 and (6.1)-(6.12) hold,

$$\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))} \leq C + C\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))}^{\frac{(1-\lambda)(\gamma-1)(4\zeta-\gamma\zeta)}{(2-\gamma)}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}} + C\|Du^\epsilon\|_{L^F([0,T];L^G(\mathbb{T}^d))}^{\frac{(1-\lambda)(\gamma-1)(2+\gamma\zeta)}{\gamma}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}}$$

Also,

$$\frac{(1-\lambda)(\gamma-1)(4\zeta-\gamma\zeta)}{(2-\gamma)}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}<1 \tag{6.16}$$

$$\frac{(1-\lambda)(\gamma-1)(2+\gamma\zeta)}{\gamma}\frac{r\upsilon\alpha\left(1-\frac{1}{\theta}\right)}{\beta_0(\theta-1)}<1 \tag{6.17}$$

have to be satisfied. The Lemma follows by combining Young's inequality with Lemma **??**. ■

To finish this chapter we present the following Theorem.

Theorem 6.0.7 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-10 are in force. Then, for any $\beta > 1$, $\|m^\epsilon\|_{L^\infty([0,T];L^\beta(\mathbb{T}^d))}$ is bounded uniformly in ϵ.*

Proof. For $p > \frac{d}{2}$, $\theta > 1$ and r is given by Lemma 6.0.6, we have by Theorem 1.1 that for any $\beta > 1$ there is r_β such that

$$\int_{\mathbb{T}^d} m^\beta(\tau,x)dt \leq C + C\||D_pH(x,Du)|^2\|^{r_\beta}_{L^r([0,T];L^p(\mathbb{T}^d))}.$$

If we combine (6.10) and Lemma 6.0.6 with A8 one obtains

$$\||D_pH(x,Du)|^2\|_{L^r([0,T];L^p(\mathbb{T}^d))} \leq C\|Du\|^{2(\gamma-1)(1-\lambda)}_{L^F([0,T];L^G(\mathbb{T}^d))} + C \leq C.$$

It is enough to conclude the Theorem. ■

Corollary 6.0.3 *Let (u^ϵ, m^ϵ) be a solution of (1.7) and assume A1-10 are in force. Then, for any $p, r > 1$, $\|Du^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))}$, $\|D^2u^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))}$ are bounded uniformly in ϵ.*

Proof. Because of Theorem 6.0.7, for $p, r > 1$, $\|g_\epsilon(m^\epsilon)\|_{L^r([0,T];L^p(\mathbb{T}^d))}$ is bounded uniformly in ϵ. So are $\|u^\epsilon\|L^\infty([0,T];L^\infty(\mathbb{T}^d))$ and $\|D^2u^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))}$ bounded by Proposition 4.0.1 and Theorem 1.3.2, respectively. Finally by Gagliardo-Nirenberg inequality

$$\|Du^\epsilon\|_{L^{2r}([0,T];L^{2p}(\mathbb{T}^d))} \leq C\|D^2u\epsilon\|^{\frac{1}{2}}_{L^r([0,T];L^p(\mathbb{T}^d))}\|u\epsilon\|^{\frac{1}{2}}_{L^\infty([0,T];L^\infty(\mathbb{T}^d))}.$$

which shows that $\|Du^\epsilon\|_{L^r([0,T];L^p(\mathbb{T}^d))}$ is also uniformly bounded in ϵ. ■

7
Lipschitz Regularity

In this chapter we derive Lipschitz regularity for the solution of u^ϵ by using the Adjoint Method, for more details about this method we encourage the readers to see [6].

Theorem 7.0.1 *Let* (u^ϵ, m^ϵ) *be a solution of (1.7) and assume A1-10 are in force. Then* $Du^\epsilon \in L^\infty(\mathbb{T}^d \times [0,T])$*, uniformly in* ϵ*. As before, we omit the* ϵ *in the proof in this Chapter.*

Proof.

$$\begin{cases} u_t + \Delta u = f \\ u(x,T) = \psi, \end{cases} \tag{7.1}$$

with $\psi \in W^{1,\infty}(\mathbb{T}^d)$ and $f \in L^a([0,T] \times \mathbb{T}^d)$ for any $a > 1$. We introduce the adjoint equation

$$-\rho_t + \Delta \rho = 0, \tag{7.2}$$

with initial data $\rho(\cdot, \tau) = \delta_{x_0}$. Multiplying (7.2) by $\nu\rho^{\nu-1}$ and integrating, we have for $\tau < s < T$. First, note that

$$\int_s^T = \nu\rho^{\rho-1}\rho_t dt = \int_s^T \frac{d}{dt}\rho^\nu dt = \rho^\nu(x,T) - \rho^\nu(x,s),$$

and

$$\int_s^T \int_{\mathbb{T}^d} \nu\rho^{\nu-1}\Delta\rho dx dt = -\nu \int_s^T \int_{\mathbb{T}^d} D\rho^{\nu-1} D\rho dx dt = \nu(1-\nu) \int_s^T \int_{\mathbb{T}^d} \rho^{\nu-2}(D\rho)^2 dx dt.$$

Observe that,

$$\frac{4(1-\nu)}{\nu}(D\rho^{\frac{\nu}{2}})^2 = \frac{4(1-\nu)}{\nu}\frac{\nu^2}{4}\rho^{\nu-2}(D\rho)^2 = \nu(1-\nu)\rho^{\nu-2}(D\rho)^2.$$

Then,

$$\int_{\mathbb{T}^d} (\rho^\nu(x,T) - \rho^\nu(x,s))dx = \frac{4(1-\nu)}{\nu} \int_s^T \int_{\mathbb{T}^d} |D\rho^{\frac{\nu}{2}}|^2 dxdt. \tag{7.3}$$

Because $\rho(\cdot,t)$ is a probability measure and $0 < \nu < 1$ we have

$$\int_{\mathbb{T}^d} \rho^\nu(x,t)dx \leq 1.$$

Thus,

$$\int_\tau^T \int_{\mathbb{T}^d} |D\rho^{\frac{\nu}{2}}|^2 dxdt \leq \frac{\nu}{4(1-\nu)}.$$

Fixing a unit vector $\xi \in \mathbb{R}^d$, we have by derivate the first equation of 7.1 in direction ξ, multiply by ρ and integrate we get

$$\int_\tau^T \int_{\mathbb{T}^d} (u_\xi)_t \rho dxdt + \int_\tau^T \int_{\mathbb{T}^d} (\Delta u_\xi)\rho dxdt = \int_\tau^T \int_{\mathbb{T}^d} f_\xi \rho dxdt,$$

multiplying (7.2) by u_ξ and subtract these equation to obtain

$$\int_\tau^T \int_{\mathbb{T}^d} (u_\xi)_t\rho - u_\xi \rho_t dxdt + \int_\tau^T \int_{\mathbb{T}^d} (\Delta u_\xi)\rho - (\Delta\rho)u_\xi dxdt = -\int_\tau^T \int_{\mathbb{T}^d} f_\xi \rho dxdt,$$

where

$$\begin{aligned}\int_\tau^T \int_{\mathbb{T}^d} (u_\xi)_t\rho - u_\xi\rho_t dxdt &= \int_\tau^T \int_{\mathbb{T}^d} \frac{d}{dt}(u_\xi \rho)dxdt = \int_{\mathbb{T}^d} (u_\xi(x,\tau)\rho(x,\tau) - u_\xi(x,T)\rho(x,T))dx \\ &= u_\xi(x,\tau)\delta_{x_0} - \int_{\mathbb{T}^d} \psi_\xi \rho(x,T)dx\end{aligned}$$

and

$$\int_\tau^T \int_{\mathbb{T}^d} (\Delta u_\xi)\rho - (\Delta\rho)u_\xi dxdt = 0.$$

Then

$$u_\xi(x,\tau)\delta_{x_0} - \int_{\mathbb{T}^d} \psi_\xi \rho(x,T)dx = -\int_\tau^T \int_{\mathbb{T}^d} f_\xi \rho dxdt = \int_\tau^T \int_{\mathbb{T}^d} f \rho_\xi dxdt.$$

Note that $|\int_{\mathbb{T}^d} \psi\rho(x,T)|dx \leq \|\psi\|_{W^{1,\infty}(\mathbb{T}^d)}$. For $0 < \nu < 1$,

$$\begin{aligned}\left|\int_\tau^T \int_{\mathbb{T}^d} f\rho_\xi dxdt\right| &\leq \int_\tau^T \int_{\mathbb{T}^d} |f|\rho^{1-\frac{\nu}{2}}|\rho^{\frac{\nu}{2}-1} D\rho| dxdt \\ &\leq \|f\|_{L^a([\tau,T]\times\mathbb{T}^d)} \|\rho^{1-\frac{\nu}{2}}\|_{L^b([\tau,T]\times\mathbb{T}^d)} \|D\rho^{\frac{\nu}{2}}\|_{L^2([\tau,T]\times\mathbb{T}^d)},\end{aligned}$$

for any $2 \leq a, b\infty$ satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{2} = 1$. Therefore it suffices to bound

$\|\rho^{1-\frac{\nu}{2}}\|_{L^b([\tau,T]\times\mathbb{T}^d)}$, for some $b > 2$.

Let $\frac{d-1}{d} < \nu < 1$, and $\kappa = \frac{d\nu}{d\nu+2}$. Then $1 - \kappa + \frac{2\kappa}{2^*\nu} = \frac{\kappa}{\nu}$, and therefore $1 < \frac{\nu}{\kappa} < \frac{2^*\nu}{2}$. Moreover $\frac{\nu}{\kappa} > 2-\nu$. Define $b = \frac{\nu}{\kappa(1-\frac{\nu}{2})} > 2$. By Holder's inequality we have

$$\left(\int_{\mathbb{T}^d} \rho^{b(1-\frac{\nu}{2})}\right)^{\frac{1}{b(1-\frac{\nu}{2})}} \leq \left(\int_{\mathbb{T}^d} \rho^{b(1-\frac{\nu}{2})}\right)^{\frac{\kappa}{\nu}} \leq \left(\int_{\mathbb{T}^d} \rho\right)^{1-\kappa} \left(\int_{\mathbb{T}^d} \rho^{\frac{2^*\nu}{2}}\right)^{\frac{2\kappa}{2^*\nu}}.$$

Recall that by Sobolev's inequality we have $\left(\rho^{\frac{2^*\nu}{2}}\right)^{\frac{2}{2^*}} \leq C + C\int_{\mathbb{T}^d} |Dp^{\frac{\nu}{2}}|^2$. Therefore

$$\int_{\mathbb{T}^d} \rho^{b(1-\frac{\nu}{2})} \leq C + C\int_{\mathbb{T}^d} |Dp^{\frac{\nu}{2}}|^2,$$

and then

$$\int_\tau^T \int_{\mathbb{T}^d} \rho^{b(1-\frac{\nu}{2})} \leq C + C\int_\tau^T \int_{\mathbb{T}^d} |Dp^{\frac{\nu}{2}}|^2 \leq C.$$

■

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