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Apêndice A
Análise de Singularidades nos Termos Presentes na Modelagem da Abertura

Ao calcularem-se os elementos da matriz C para a cavidade via equação (3.29 c), deve-se notar a possível presença de singularidades no integrando da função integral. Isto ocorre quando o ponto de observação está localizado na região da fonte.

Neste apêndice cada integrando será investigado a fim de determinar se há realmente a singularidade, e no caso afirmativo, um artifício numérico será utilizado.

Reescrevendo aqui equação (4.38):

$$\overline{H}^{ext}\left(\overline{M}\right)=-\frac{j\omega\varepsilon_r}{2\pi}\iint_{S'}\overline{M}\left(\overline{r}'\right)G\left(\overline{r},\overline{r}'\right)dS'+\frac{1}{2\pi\mu}\nabla\iint_{S'}\nabla'\cdot\overline{M}\left(\overline{r}'\right)G\left(\overline{r},\overline{r}'\right)dS' \quad (4.38)$$

Onde chamaremos de $\overline{H_1}^{ext}\left(\overline{M}\right)$ e $\overline{H_2}^{ext}\left(\overline{M}\right)$ as seguintes equações (A.1 a) e (A.1 b):

$$\overline{H_1}^{ext}\left(\overline{M}\right)=-\frac{j\omega\varepsilon_r}{2\pi}\iint_{S'}\overline{M}\left(\overline{r}'\right)G\left(\overline{r},\overline{r}'\right)dS' \quad \text{(A.1 a)}$$

$$\overline{H_2}^{ext}\left(\overline{M}\right)=\frac{1}{2\pi\mu}\nabla\iint_{S'}\nabla'\cdot\overline{M}\left(\overline{r}'\right)G\left(\overline{r},\overline{r}'\right)dS' \quad \text{(A.1 b)}$$

Sabendo-se que $G\left(\overline{r},\overline{r}'\right)=\frac{e^{-jk_iR}}{R}$, e substituindo na equação (A.1) tem-se:

$$\overline{H_1}^{ext}\left(\overline{M}\right)=-\frac{j\omega\varepsilon_r}{2\pi}\iint_{S'}\overline{M}\left(\overline{r}'\right)\frac{e^{-jk_iR}}{R}dS' \quad \text{(A.2 a)}$$

$$\overline{H_2}^{ext}\left(\overline{M}\right)=\frac{1}{2\pi\mu}\nabla\iint_{S'}\nabla'\cdot\overline{M}\left(\overline{r}'\right)\frac{e^{-jk_iR}}{R}dS' \quad \text{(A.2 b)}$$

onde

$$R=\left|\overline{r}-\overline{r}'\right|=\left[\left(x-x'\right)^2+\left(y-y'\right)^2+z^2\right]^{1/2} \quad \text{(A.3)}$$

À medida que $x\rightarrow x'$ e $y\rightarrow y'$,o integrando da equação (A.2) é nitidamente singular. Primeiramente, analisando a equação (A.2 a), é conveniente reescrever-se como:

$$\overline{H_1}^{ext}(\overline{M}) = I_1 + I_2 \quad \text{(A.4)}$$

onde

$$I_1 = -\frac{j\omega\varepsilon_r}{2\pi}\iint_{S'}\overline{M}(\overline{r}')\frac{e^{-jk_iR}}{R} - \overline{M}(\overline{r}')\Big|_{\substack{x'=x_{medio}\\ y'=y_{medio}}} \cdot \frac{1}{R}\,dS' \quad \text{(A.5 a)}$$

$$I_2 = -\frac{j\omega\varepsilon_r}{2\pi}\left[\overline{M}(\overline{r}')\Big|_{\substack{x'=x_{medio}\\ y'=y_{medio}}}\iint_{S'}\frac{1}{R}\,dS'\right] \quad \text{(A.5 b)}$$

Deste modo, o integrando em I_1 não é mais singular.Falta solucionar agora o problema da singularidade em I_2. Reescrevendo (A.5 b) de maneira a destacar a singularidade tem-se:

$$I_2 = -\frac{j\omega\varepsilon_r}{2\pi}\left[\overline{M}(\overline{r}')\Big|_{\substack{x'=x_{medio}\\ y'=y_{medio}}} \cdot I'_2\right] \quad \text{(A.6)}$$

onde

$$I'_2 = \iint_{S'}\frac{1}{R}\,dS' \quad \text{(A.7)}$$

A integral (A.7) é realizada em S’, de acordo com a malha projetada para a abertura em quadriláteros, como pode ser observado pela figura 4.5.

Assim, a equação (A.7) pode ser determinada por:

$$I'_2 = \int_{y1}^{y2}\int_{x'=a43y'}^{x'=a12y'+b12}\frac{1}{\sqrt{(x'-x)^2(y'-y)^2}}\,dx'dy' \quad \text{(A.7 a)}$$

Chamando $x'' = (x'-x)$ e sabendo que $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$ e substituindo em (A.7 a):

$$I'_2 = \int_{y1}^{y2}\int_{x''=a43y'+x}^{x''=a12y'+b12+x}\frac{1}{\sqrt{x''^2+(y'-y)^2}}\,dx''dy' \quad \text{(A.7 b)}$$

Com isso, I'_2 pode ser determinado por:

$$I'_2 = \overline{I'}_2 - \overline{\overline{I'}}_2 \quad \text{(A.8)}$$

onde

$$\overline{I'}_2 = \int_{y1}^{y2}\ln\left[(a_{12}y'+b_{12}-x)+\sqrt{(a_{12}y'+b_{12}-x)^2+(y'-y)^2}\right]dy' \quad \text{(A.8 a)}$$

$$\overline{\overline{I'}}_2 = \int_{y1}^{y2}\ln\left[(a_{43}y'-x)+\sqrt{(a_{43}y'-x)^2+(y'-y)^2}\right]dy' \quad \text{(A.8 b)}$$

Concentrando-se no cálculo de (A.8 a) percebe-se a presença de singularidades quando $y \to y'$. Assim:

$$\lim_{y'\to y}\left[\left(a_{12}y'+b_{12}-x\right)+\sqrt{\left(a_{12}y'+b_{12}-x\right)^2+\left(y'-y'\right)^2}\right]=$$
$$\left(a_{12}y'+b_{12}-x\right)+\left|a_{12}y'+b_{12}-x\right|+\frac{1}{2}\frac{\left(y'-y'\right)^2}{\left|a_{12}y'+b_{12}-x\right|} \qquad \text{(A.9)}$$

E de acordo com esta equação (A.9) deve-se analisar as diversas possibilidades na determinação de (A.8 a):

$a_{12}\neq 0$

$x>a_{12}y+b_{12}$

$$\overline{I'}_2=\int_{y1}^{y2}\ln\left[\left(a_{12}y'+b_{12}-x\right)+\sqrt{\left(a_{12}y'+b_{12}-x\right)^2+\left(y'-y\right)^2}\right]-\ln\left(\frac{\left(y'-y\right)^2}{2\left|-a_{12}y'-b_{12}+x\right|}\right)dy'+$$
$$2\int_{y1}^{y2}\ln\left|y'-y\right|\,dy'-\int_{y1}^{y2}\ln\left(2\left|-a_{12}y'-b_{12}+x\right|\right)dy'$$

(A.10 a)

$x<a_{12}y+b_{12}$

$$\overline{I'}_2=\int_{y1}^{y2}\ln\left[\left(a_{12}y'+b_{12}-x\right)+\sqrt{\left(a_{12}y'+b_{12}-x\right)^2+\left(y'-y\right)^2}\right]dy' \qquad \text{(A.10 b)}$$

$x=a_{12}y+b_{12}$

$y>y1 \quad e \quad y>y2$

$$\overline{I'}_2=\ln\left(\sqrt{a_{12}^{\ 2}+1}-a_{12}\right)\left(y2-y1\right)+\left(y2-y\right)\ln\left(\left|y2-y\right|\right)+$$
$$\left(y-y1\right)\ln\left(y-y1\right)-\left(y2-y1\right) \qquad \text{(A.10 c)}$$

$y<y1 \quad e \quad y<y2$

$$\overline{I'}_2=\ln\left(\sqrt{a_{12}^{\ 2}+1}+a_{12}\right)\left(y2-y1\right)+$$
$$\left(y2-y\right)\ln\left(\left|y2-y\right|\right)+\left(y-y1\right)\ln\left(y-y1\right)-\left(y2-y1\right) \qquad \text{(A.10 d)}$$

$y1<y<y2$

$$\overline{I'}_2=\ln\left(\sqrt{a_{12}^{\ 2}+1}-a_{12}\right)\left(y-y1\right)+\ln\left(\sqrt{a_{12}^{\ 2}+1}+a_{12}\right)\left(y2-y\right)+$$
$$\left(y2-y\right)\ln\left(\left|y2-y\right|\right)+\left(y-y1\right)\ln\left(y-y1\right)-\left(y2-y1\right) \qquad \text{(A.10 e)}$$

$a_{12}=0$

$b_{12} \neq 0$

$$\overline{I'}_2 = \int_{y1}^{y2} \ln \left[(b_{12} - x) + \sqrt{(b_{12} - x)^2 + (y' - y)^2} \right] dy' \quad \text{(A.11)}$$

Concentrando-se no cálculo de (A.11 a) percebe-se a presença de singularidades quando $y \to y'$. Assim:

$$\begin{aligned} &\lim_{y' \to y} \left[(b_{12} - x) + \sqrt{(b_{12} - x)^2 + (y' - y')^2} \right] = \\ &\quad (b_{12} - x) + |b_{12} - x| + \frac{1}{2} \frac{(y' - y')^2}{|b_{12} - x|} \end{aligned} \quad \text{(A.12)}$$

$b_{12} < 0$

$$\overline{I'}_2 = \int_{y1}^{y2} \ln \left[(b_{12} - x) + \sqrt{(b_{12} - x)^2 + (y' - y)^2} \right] dy' \quad \text{(A.13 a)}$$

$b_{12} > 0$

$$\begin{aligned} \overline{I'}_2 = &\int_{y1}^{y2} \ln \left[(b_{12} - x) + \sqrt{(b_{12} - x)^2 + (y' - y)^2} \right] - \ln \left(\frac{(y' - y)^2}{2\,|-b_{12} + x|} \right) dy' + \\ &2 \int_{y1}^{y2} \ln |y' - y| \, dy' - \int_{y1}^{y2} \ln (2 |-b_{12} + x|) \, dy' \end{aligned} \quad \text{(A.13 b)}$$

$x = b_{12}$

$$\overline{I'}_2 = 2 \int_{y1}^{y2} \ln |y' - y| \, dy' \quad \text{(A.13 c)}$$

Analogamente, na determinação de (A.8 b) percebe-se a presença de singularidades quando $y \to y'$. Assim:

$$\begin{aligned} &\lim_{y' \to y} \left[(a_{43} y' - x) + \sqrt{(a_{43} y' - x)^2 + (y' - y')^2} \right] = \\ &\quad (a_{43} y' - x) + |a_{43} y' - x| + \frac{1}{2} \frac{(y' - y')^2}{|a_{43} y' - x|} \end{aligned} \quad \text{(A.14)}$$

E de acordo com esta equação (A.14) deve-se analisar as diversas possibilidades na determinação de (A.8 b):

$a_{43} \neq 0$

$x > a_{43} y$

$$\overline{\overline{I}}'_2 = \int_{y1}^{y2} \ln\left[\left(a_{43}y'-x\right)+\sqrt{\left(a_{43}y'-x\right)^2+\left(y'-y\right)^2}\right] - \ln\left(\frac{\left(y'-y\right)^2}{2\left|-a_{43}y'+x\right|}\right)dy'+ \\ 2\int_{y1}^{y2}\ln\left|y'-y\right|dy' - \int_{y1}^{y2}\ln\left(2\left|-a_{43}y'+x\right|\right)dy' \quad \text{(A.15 a)}$$

$x < a_{43}y$

$$\overline{\overline{I}}'_2 = \int_{y1}^{y2} \ln\left[\left(a_{43}y'-x\right)+\sqrt{\left(a_{43}y'-x\right)^2+\left(y'-y\right)^2}\right]dy' \quad \text{(A.15 b)}$$

$x = a_{43}y$

$y > y1 \quad e \quad y > y2$

$$\overline{\overline{I}}'_2 = \ln\left(\sqrt{a_{43}^{\ 2}+1}-a_{43}\right)\left(y2-y1\right)+\left(y2-y\right)\ln\left(\left|y2-y\right|\right)+ \\ \left(y-y1\right)\ln\left(y-y1\right)-\left(y2-y1\right) \quad \text{(A.15 c)}$$

$y < y1 \quad e \quad y < y2$

$$\overline{I}'_2 = \ln\left(\sqrt{a_{43}^{\ 2}+1}+a_{43}\right)\left(y2-y1\right)+ \\ \left(y2-y\right)\ln\left(\left|y2-y\right|\right)+\left(y-y1\right)\ln\left(y-y1\right)-\left(y2-y1\right) \quad \text{(A.15 d)}$$

$y1 < y < y2$

$$\overline{I}'_2 = \ln\left(\sqrt{a_{43}^{\ 2}+1}-a_{43}\right)\left(y-y1\right)+\ln\left(\sqrt{a_{43}^{\ 2}+1}+a_{43}\right)\left(y2-y\right)+ \\ \left(y2-y\right)\ln\left(\left|y2-y\right|\right)+\left(y-y1\right)\ln\left(y-y1\right)-\left(y2-y1\right) \quad \text{(A.15 e)}$$

$a_{43} = 0$

$x \neq 0$

$$\overline{\overline{I}}'_2 = \int_{y1}^{y2} \ln\left[\left(-x\right)+\sqrt{\left(-x\right)^2+\left(y'-y\right)^2}\right]dy' \quad \text{(A.16)}$$

Observando a equação (A.16) deve-se notar as possíveis singularidades respectivas a função logaritmo, e isto ocorre quando $y \rightarrow y'$. Assim:

$$\lim_{y'\rightarrow y}\left[\left(-x\right)+\sqrt{\left(-x\right)^2+\left(y'-y'\right)^2}\right] = \\ \left(-x\right)+\left|-x\right|+\frac{1}{2}\frac{\left(y'-y'\right)^2}{\left|-x\right|} \quad \text{(A.17)}$$

$x > 0$

$$\overline{I'_2} = \int_{y1}^{y2} \ln\left[(-x) + \sqrt{(-x)^2 + (y'-y)^2}\right] - \ln\left(\frac{(y'-y)^2}{2\,|+x|}\right) dy' + \qquad \text{(A.18 a)}$$
$$2\int_{y1}^{y2} \ln|y'-y|\,dy' - \int_{y1}^{y2} \ln\left(2|+x|\right) dy'$$

$x < 0$

$$\overline{\overline{I'_2}} = \int_{y1}^{y2} \ln\left[(-x) + \sqrt{(-x)^2 + (y'-y)^2}\right] dy' \quad \text{(A.18 b)}$$

$x = 0$

$$\overline{I'_2} = 2\int_{y1}^{y2} \ln|y'-y|\,dy' \quad \text{(A.18 c)}$$

Feito isso, a solução da equação (A.2 a) é determinada por:

$$\overline{H_1}^{ext}(\overline{M}) = -\frac{j\omega\varepsilon}{2\pi}\left\{\int\left[\overline{M}\left(\bar{r}'\right)G\left(\bar{r},\bar{r}'\right) - M\Big|_{\substack{x=xmed\\ y=ymed}} \cdot \frac{1}{R}\right] dS' + M\Big|_{\substack{x=xmed\\ y=ymed}} \int \frac{1}{R}\, dS'\right\}$$

(A.19)

Examinando a equação (A. 2 b), faz-se necessário encontrar $\nabla'\cdot\overline{M}$:

$$\overline{M} = \overline{W}^{2,4} \times \hat{z}$$
$$= \frac{c'[1 + S_E\varepsilon]}{2\sqrt{(c + S_E d)^2 + c'^2}}\,\hat{x} - \frac{(c + d\varepsilon)[1 + S_E\varepsilon]}{2\sqrt{(c + S_E d)^2 + c'^2}}\,\hat{y} \quad \text{(A.20)}$$

onde $\overline{W}^{2,4}$ = $(1 + S_\xi\xi)\left(\delta_z + S_z\,\dfrac{z}{h}\right)\dfrac{(c + d\xi)\hat{x} + c'\,\hat{y}}{2\sqrt{(c + dS_\xi)^2 + c'^2}}$

Assim:

$$\nabla'\cdot\overline{M} = \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}$$
$$= \frac{S_E c'^2}{2\sqrt{(c + S_E d)^2 + c'^2}\,\left(c'b + dy - da'\right)} + \qquad \text{(A.21)}$$
$$\frac{\left[d(1 + S_E\varepsilon) + S_E(c + d\varepsilon)\right](cb + dx - da)}{2\sqrt{(c + S_E d)^2 + c'^2}\,\left(c'b + dy - da'\right)^2}$$

Definido $\nabla'\cdot\overline{M}$, chamaremos $\int \nabla'\cdot\overline{M}\left(\bar{r}'\right)G\left(\bar{r},\bar{r}'\right)dS'$ de *GMJ(x,y)* a fim de facilitar o estudo.

Substituindo em (A. 2 b), pode-se determinar a solução desta equação da mesma maneira que a solução de (A. 1b) foi encontrada.

Desta forma, os elementos da matriz C, C_{ij}^{e}, são dadas por:

$$C_{ij}^{e} = \overline{C}_{ij}^{e} + \overline{\overline{C}}_{ij}^{e} \quad \text{(A.22)}$$

onde $\overline{C}_{ij}^{e}$ refere-se à $\overline{H_1}^{ext}(\overline{M})$ e $\overline{\overline{C}}_{ij}^{e}$ à $\overline{H}_2^{ext}(\overline{M})$.

$$\overline{C}_{ij}^{e} = \int \overline{M}_i\, \overline{H}^{ext}\left(M_j\right) dS' \quad \text{(A.23 a)}$$

$$\overline{C}_{ij}^{e} = \overline{H}_x^{ext}\left(M_j\right)\Big|_{ponto\ médio} \int\!\!\int \overline{M}_{ix}\ dS' + \overline{H}_y^{ext}\left(M_j\right)\Big|_{ponto\ médio} \int\!\!\int \overline{M}_{iy}\ dS' \quad \text{(A.23 b)}$$

Já $\overline{\overline{C}}_{ij}^{e}$, é determinada por:

$$\overline{\overline{C}}_{ij}^{e} = \nabla \int_{M_i} \overline{M}_i\ GMJ(x,y)\, dS \quad \text{(A.24 a)}$$

$$\overline{\overline{C}}_{ij}^{e} = \int_{M_i} \left\{ \overline{M}_{ix} \frac{\partial GMJ(x,y)}{\partial x} + \overline{M}_{iy} \frac{\partial GMJ(x,y)}{\partial y} \right\} dx\, dy \quad \text{(A.24 b)}$$

Integrando por partes:

$$
\begin{aligned}
\overline{\overline{C}}_{ij}^{e} = {} & \int_{y4}^{y3} \left[\overline{M}_{ix}\ GMJ(x,y)\right]_{x=a43\,ymedio+b43}^{x=a12\,ymedio+b12} dy \; - \\
& \int_{y4}^{y3} \int_{x=a43\,ymedio+b43}^{x=a12\,ymedio+b12} \left[\frac{\partial \overline{M}_{ix}}{\partial x} + \frac{\partial \overline{M}_{iy}}{\partial y} \right] GMJ(x,y)\, dx\, dy \; + \\
& \int_{\min(x3,x4)}^{\max(x3,x4)} \left[\overline{M}_{iy} GMJ(x,y)\right] \underbrace{\Big|_{y=y4}^{y=\frac{x}{a43}-b43}}_{a43>0} \underbrace{\Big|_{y=\frac{x}{a43}-b43}^{y=y3}}_{a43<0} dx \; + \\
& \int_{\max(x3,x4)}^{\min(x1,x2)} \left[\overline{M}_{iy} GMJ(x,y)\right]_{y=y1}^{y=y4} dx \; + \\
& \int_{\min(x1,x2)}^{\max(x1,x2)} \left[\overline{M}_{iy} GMJ(x,y)\right] \underbrace{\Big|_{y=y1}^{y=\frac{x}{a12}-b12}}_{a12<0} \underbrace{\Big|_{y=\frac{x}{a12}-b12}^{y=y2}}_{a12>0} dx
\end{aligned}
$$