2
Mechanics of membranes

In this chapter the basis for the numerical analysis of membrane structures are presented.

2.1
Kinematics

Kinematics is the study of the deformation and motion of a continuous body. This body in an initial state is shown in figure 2.1 with number 1. Successive deformations are applied in this body represented with the numbers 2 and 3. The reference configuration in Lagrangian description is defined in the state 1 and the states 2 and 3 are the current configuration. In Eulerian description the reference configuration is updated. For example, in the first applied deformation the reference configuration is the state 1 and the current configuration is the state 2 in the second applied deformation, the state 2 becomes the reference configuration and the state 3 is the current configuration. In the present work the Lagrangian description is adopted in the implementation.

The deformation gradient **F** transforms the reference configuration into the actual configuration.

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{2-1}$$

where **x** is the position of a point in current configuration and **X** is the position of a point in the reference configuration.

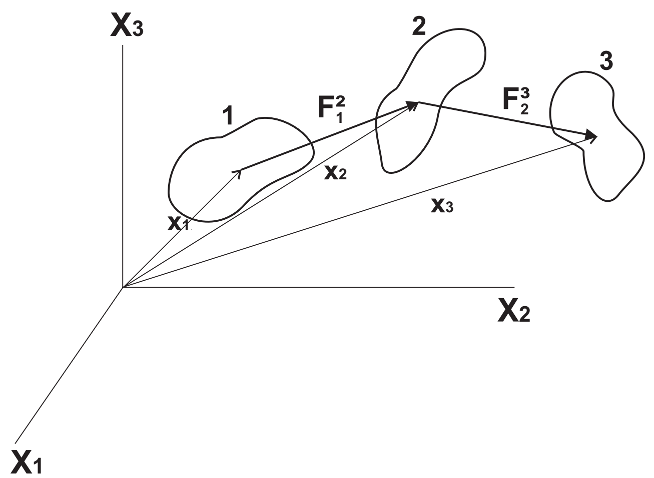

Figure 2.1: Successive deformations of a continuous body

According to Lee and Liu [38], the combination of elastic and plastic strains, both finite, calls for a more careful study of the kinematics than the usual assumption that the total strain components are simply the sum of the elastic and plastic components, as for infinitesimal strain theory.

This hypothesis was introduced by Lee and Liu [38] and is defined as the product:

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{2-2}$$

The transformation from the first position to the second position is given by:

$$d\mathbf{x}_2 = \mathbf{F}_1^2 d\mathbf{x}_1 \tag{2-3}$$

where $\mathbf{x}_1(X_1, X_2, X_3)$ and $\mathbf{x}_2(X_1, X_2, X_3)$ are the coordinates for the undeformed body (first position) and deformed body (second position), respectively.

Similarly, the transformation from the first position to the third position is:

$$d\mathbf{x}_3 = \mathbf{F}_1^3 d\mathbf{x}_1 \tag{2-4}$$

and the second position to the third position:

$$d\mathbf{x}_3 = \mathbf{F}_2^3 d\mathbf{x}_2 = \mathbf{F}_2^3\mathbf{F}_1^2 d\mathbf{x}_1 \tag{2-5}$$

Substituting equation 2-4 in 2-5 results:

$$\mathbf{F}_1^3 = \mathbf{F}_2^3\mathbf{F}_1^2 \tag{2-6}$$

According to Lee and Liu [38], such transformations provide a convenient means of representing elastoplastic deformations in the neighborhood of a particle. If the stress in the final configuration is removed and the temperature reduced to the uniform initial temperature, the elastic and thermal deformations will be recovered, leaving only permanent plastic deformations which provide the second configuration. Therefore, equation 2-6 results in equation 2-2 and it can be represented according to Simo and Hughes [39], Souza Neto et al. [40], Simo and Ortiz [41], and Simo ([42],[43]) by figure 2.2.

2.2 Strain measure

Strain express the geometrical deformation and motion of a body. In Lagrangian description the Green-Lagrange strain tensor is defined by:

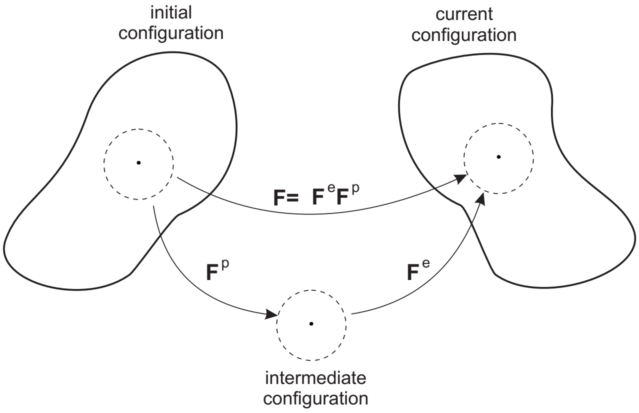

Figure 2.2: Multiplicative decomposition of the deformation gradient (source: Souza Neto et al. [40])

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \tag{2-7}$$

The logarithmic strain measure in Lagrangian description is defined:

$$\mathbf{E}_L = ln(\mathbf{U}) \tag{2-8}$$

where $\mathbf{U}$ is termed the right stretch tensor.

$$\mathbf{U} = \sqrt{\mathbf{C}} \tag{2-9}$$

where $\mathbf{C}$ is the right Cauchy-Green tensor and its spectral representation is given by:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 = \sum_{i=0}^{m} \lambda_i \mathbf{M}_i \qquad i = 1, 2 \tag{2-10}$$

where λ_i are the principal stretches and $\mathbf{M}_i$ are the eigenprojections.

With the eigenprojections, the values $cos^2\phi$, $sin^2\phi$ and, $cos\phi sin\phi$ are obtained:

$$\mathbf{M}_1 = \begin{bmatrix} cos^2\phi & cos\phi sin\phi \\ cos\phi sin\phi & sin^2\phi \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} sin^2\phi & -cos\phi sin\phi \\ -cos\phi sin\phi & cos^2\phi \end{bmatrix} \tag{2-11}$$

Equation 2-8 is rewritten in spectral representation:

$$\mathbf{E}_L = \sum_{i=0}^{2} E_{Li}\mathbf{M}_i = \sum_{i=0}^{2} ln(\lambda_i)\mathbf{M}_i \qquad i = 1, 2 \tag{2-12}$$

2.3 Stress measure

Force per unit area physically express stress measure. This measure rise from the forces of a body due to the their deformation and motion. The conjugated stress pair with Green-Lagrange strain tensor is the second Piola-Kirchhoff stress tensor, given by:

$$\mathbf{S} = \mathbf{P}\mathbf{F}^{-T} \tag{2-13}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, measured with force per unit area defined in the reference configuration.

The first Piola-Kirchhoff stress tensor is not symmetric. Therefore the second Piola-Kirchhoff stress tensor is often used, which is symmetric but does not admit a physical interpretation in terms of surface traction.

The Kirchhoff stress ($\mathbf{T}$) in Lagrangian description conjugate with the logarithmic strain in Lagrangian description and it is related with the Kirchhoff stress in Eulerian description ($\boldsymbol{\tau}$) with:

$$\mathbf{T} = \mathbf{R}^T\boldsymbol{\tau}\mathbf{R} \tag{2-14}$$

The relation between the Kirchhoff stress tensor in Eulerian description and the second Piola-Kirchhoff stress tensor is given by:

$$\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^T \tag{2-15}$$

2.4 Membrane formulation

Otto[27, 1] defines a membrane as a flexible skin stretched in such a way to be subjected to tension.

The membrane formulation presented here is taken from works of Wüchner and Bletzinger [44], Vázquez [35], Holzapfel [45] and Linhard [31].

A point on the surface in the reference configuration (Ω_0) is described by a position vector $\mathbf{X}$ which depends on two independent surface coordinates ξ^1 and ξ^2, presented in Figure 2.3.

$$\mathbf{X} = \mathbf{X}(\xi^1, \xi^2) \tag{2-16}$$

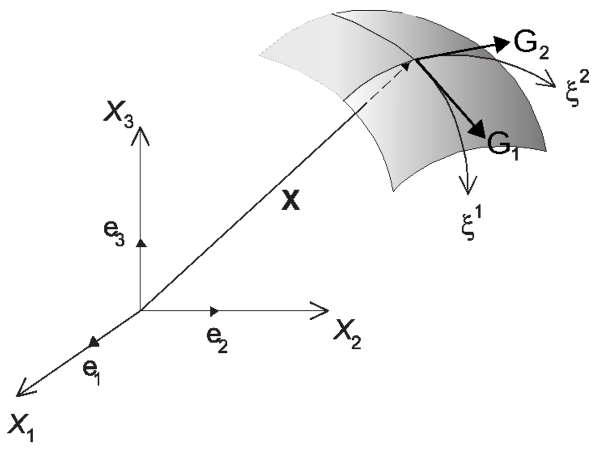

Figure 2.3: Membrane coordinates

The position vector **x** in the current configuration is defined by:

$$\mathbf{x} = \mathbf{x}(\xi^1, \xi^2) \tag{2-17}$$

The covariant base vectors in the reference and current configuration are defined respectively by the differentiation of **X** and **x** with respect to ξ^1 and ξ^2:

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{X}}{\partial \xi^\alpha}, \qquad \mathbf{g}_\alpha = \frac{\partial \mathbf{x}}{\partial \xi^\alpha}, \qquad \alpha = 1, 2 \tag{2-18}$$

The covariant base vectors are tangential to the corresponding coordinate lines. The surface normals are determined by **N** or **n**, defined through the normalized cross product:

$$\mathbf{G}_3 = \mathbf{G}_1 \times \mathbf{G}_2, \quad \mathbf{N} = \frac{\mathbf{G}_3}{\|\mathbf{G}_3\|} \qquad \mathbf{g}_3 = \mathbf{g}_1 \times \mathbf{g}_2, \quad \mathbf{n} = \frac{\mathbf{g}_3}{\|\mathbf{g}_3\|} \tag{2-19}$$

The covariant metric tensors are

$$G_{\alpha\beta} = \mathbf{G}_\alpha \cdot \mathbf{G}_\beta \qquad g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta \tag{2-20}$$

The contravariant base vectors $\mathbf{G}^\alpha$ and $\mathbf{g}^\alpha$ are given by

$$\mathbf{G}^\alpha = G^{\alpha\beta} \cdot \mathbf{G}_\beta \qquad \mathbf{g}^\alpha = g^{\alpha\beta} \cdot \mathbf{g}_\beta \tag{2-21}$$

where the contravariant metric tensors are

$$G^{\alpha\beta} = {G_{\alpha\beta}}^{-1} \qquad g^{\alpha\beta} = {g_{\alpha\beta}}^{-1} \tag{2-22}$$

The relations between the covariant and contravariant base vectors are given

$$\mathbf{G}^{\alpha} \cdot \mathbf{G}_{\beta} = \delta^{\alpha}_{\beta} \qquad \mathbf{g}^{\alpha} \cdot \mathbf{g}_{\beta} = \delta^{\alpha}_{\beta} \tag{2-23}$$

where the Kronecker delta is:

$$\delta^{\alpha}_{\beta} = \begin{cases} 1 & if \;\; \alpha = \beta \\ 0 & otherwise \end{cases} \tag{2-24}$$

The deformation gradient **F** in curvilinear coordinates is given by:

$$\mathbf{F} = \mathbf{g}_{\alpha} \otimes \mathbf{G}^{\alpha}; \quad \mathbf{F}^{T} = \mathbf{G}^{\alpha} \otimes \mathbf{g}_{\alpha}; \quad \mathbf{F}^{-1} = \mathbf{G}_{\alpha} \otimes \mathbf{g}^{\alpha}; \quad \mathbf{F}^{-T} = \mathbf{g}^{\alpha} \otimes \mathbf{G}_{\alpha} \tag{2-25}$$

The Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor are defined as:

$$\mathbf{E} = \frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right)\mathbf{G}^{\alpha} \otimes \mathbf{G}^{\beta} \tag{2-26}$$

$$\mathbf{S} = S^{\alpha\beta}\mathbf{G}_{\alpha} \otimes \mathbf{G}_{\beta} \tag{2-27}$$

The second Piola-Kirchhoff stress tensor is obtained from a constitutive relation with the Green-Lagrange strain tensor.

2.4.1
Finite element discretization

The finite element discretization is developed with shape functions in terms of isoparametric coordinates (ξ^1, ξ^2) for the total Lagrangian formulation. Hence the position vectors for the reference and current configuration are expressed by:

$$\mathbf{X}(\xi^1, \xi^2) = \sum_{i}^{n_{node}} N_i(\xi^1, \xi^2)\mathbf{X}_i \qquad \mathbf{x}(\xi^1, \xi^2) = \sum_{i}^{n_{node}} N_i(\xi^1, \xi^2)\mathbf{x}_i \tag{2-28}$$

where N_i are the shape functions.

Replacing equation 2-28 in equation 2-18 gives:

$$\mathbf{G}_{\alpha} = \frac{\partial\left(\sum_{i}^{n_{node}} N_i(\xi^1, \xi^2)\mathbf{X}_i\right)}{\partial\xi^{\alpha}} = \sum_{i}^{n_{node}} \frac{\partial N_i(\xi^1, \xi^2)}{\partial\xi^{\alpha}}\mathbf{X}_i \tag{2-29}$$

$$\mathbf{g}_{\alpha} = \frac{\partial\left(\sum_{i}^{n_{node}} N_i(\xi^1, \xi^2)\mathbf{x}_i\right)}{\partial\xi^{\alpha}} = \sum_{i}^{n_{node}} \frac{\partial N_i(\xi^1, \xi^2)}{\partial\xi^{\alpha}}\mathbf{x}_i$$

2.4.2
Linearization of the virtual work

The virtual work principle is used to establish the equilibrium conditions for the static analysis. This principle will be briefly described. For more details see Zienkiewicz [46] and Bathe [47].

The virtual work principle states that the equilibrium of a body requires that for any compatible small virtual displacements imposed on the body in its state of equilibrium, the total internal virtual work is equal to the total external virtual work:

$$\int_V \delta\varepsilon \cdot \sigma \, dV = \int_V \delta U \cdot f^B \, dV + \int_V \delta U^S \cdot f^S \, dS + \sum_i \delta U^i \cdot R_c^i \qquad \text{(2-30)}$$

$$\delta W_{int} = \delta W_{ext}$$

where ε are virtual strains corresponding to virtual displacements U, σ are the stresses in equilibrium with applied loads, f^B are applied body forces, f^S are applied surface forces and R_C are concentrated loads.

The internal virtual work (δW_{int}) is linearized for the solution with a Newton scheme. Therefore, the left-hand-side of equation 2-30 is expanded into a Taylor series up to the first order terms:

$$\delta W_{int}{}^{lin} = \int_V (\delta\mathbf{E} \cdot \mathbf{S} + \Delta\delta\mathbf{E} \cdot \mathbf{S} + \delta\mathbf{E} \cdot \Delta\mathbf{S}) \, dV \qquad \text{(2-31)}$$

To obtain approximated solutions in a form suitable for finite element analysis the variation principle is established. The finite element equations derived are simply the statements of this variation with respect to displacements:

$$\frac{\partial W}{\partial u_i} = 0 \qquad \text{(2-32)}$$

Substituting equations 2-26 and 2-27 into the equation 2-31 gives:

$$\begin{aligned} \delta W_{int}{}^{lin} = & \int_V \left(\delta\left(\frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right)\mathbf{G}^\alpha \otimes \mathbf{G}^\beta\right) \cdot \mathbf{S}\right) dV + \\ & \int_V \left(\Delta\delta\left(\frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right)\mathbf{G}^\alpha \otimes \mathbf{G}^\beta\right) \cdot \mathbf{S}\right) dV + \\ & \int_V \left(\delta\left(\frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right)\mathbf{G}^\alpha \otimes \mathbf{G}^\beta\right) \cdot \Delta(\mathbf{S})\right) dV \end{aligned} \qquad \text{(2-33)}$$

Applying the variational principle (equation 2-32):

$$\frac{\partial W_{int}^{lin}}{\partial u_i} = h \int_A \frac{\delta\mathbf{E}}{\partial u_i}\mathbf{S} \, dA + h \int_A \left(\frac{\partial\delta\mathbf{E}}{\partial u_j}\mathbf{S} + \delta\mathbf{E}\frac{\partial\mathbf{S}}{\partial u_j}\right) dA = 0 \qquad \text{(2-34)}$$

where h is the membrane thickness and A is the membrane surface area.

$$\mathbf{f}_{int} = h \int_A \frac{\delta \mathbf{E}}{\partial u_i} \mathbf{S}\, dA \tag{2-35}$$

$$\mathbf{K}_T = h \int_A \left(\frac{\partial \delta \mathbf{E}}{\partial u_j} \mathbf{S} \right) dA + h \int_A \left(\delta \mathbf{E} \frac{\partial \mathbf{S}}{\partial u_j} \right) dA \tag{2-36}$$

where $\delta\mathbf{E}$ is derived w.r.t δu_i:

$$\begin{aligned} \frac{\delta \mathbf{E}}{\delta u_i} &= \frac{\delta \left(\frac{1}{2} \left(g_{\alpha\beta} - G_{\alpha\beta} \right) G^{\alpha} \otimes G^{\beta} \right)}{\delta u_i} = \frac{1}{2} \cdot \left(\frac{\delta g_{\alpha\beta}}{\delta u_i} \right) G^{\alpha} \otimes G^{\beta} \\ &= \frac{1}{2} \cdot \left(\frac{\delta g_{\alpha} g_{\beta}}{\delta u_i} \right) G^{\alpha} \otimes G^{\beta} = \frac{1}{2} \cdot \left(\frac{\delta g_{\alpha}}{\delta u_i} g_{\beta} + g_{\alpha} \frac{\delta g_{\beta}}{\delta u_i} \right) G^{\alpha} \otimes G^{\beta} \end{aligned} \tag{2-37}$$

The equation for the internal forces is given by:

$$\mathbf{f}_{int} = h \cdot \int_A \left(\frac{1}{2} \left(\frac{\partial g_{\alpha}}{\partial u_i} g_{\beta} + g_{\alpha} \frac{\partial g_{\beta}}{\partial u_i} \right) G^{\alpha} \otimes G^{\beta} \right) S^{\alpha\beta} G_{\alpha} \otimes G_{\beta} \;\; dA \tag{2-38}$$

where $\frac{\delta g_{\alpha}}{\delta u_i}$ and $\frac{\delta g_{\beta}}{\delta u_i}$ are:

$$\frac{\delta g_{\alpha}}{\delta u_i} = \frac{\partial g_{\alpha}}{\partial u_i} \delta u_i \qquad \frac{\delta g_{\beta}}{\delta u_i} = \frac{\partial g_{\beta}}{\partial u_i} \delta u_i \tag{2-39}$$

The first term of the stiffness matrix (equation 2-36) is obtained through the equation:

$$\begin{aligned} \frac{\partial \delta \mathbf{E}}{\partial u_j} &= \frac{\partial \left(\frac{1}{2} \left(\frac{\partial g_{\alpha}}{\partial u_i} \cdot g_{\beta} + g_{\alpha} \cdot \frac{\partial g_{\beta}}{\partial u_i} \right) \right)}{\partial u_j} \\ &= \frac{1}{2} \left(\frac{\partial^2 g_{\alpha}}{\partial u_i \partial u_j} + \frac{\partial g_{\alpha}}{\partial u_i} \frac{\partial g_{\beta}}{\partial u_j} + \frac{\partial g_{\alpha}}{\partial u_j} \frac{\partial g_{\beta}}{\partial u_i} + \frac{\partial^2 g_{\beta}}{\partial u_i \partial u_j} \right) \end{aligned} \tag{2-40}$$

the second derivatives vanish:

$$\frac{\partial^2 g_{\alpha}}{\partial u_i \partial u_j} = 0 \qquad \frac{\partial^2 g_{\beta}}{\partial u_i \partial u_j} = 0 \tag{2-41}$$

Substituting equation 2-40 in the first term of equation 2-36 gives:

$$\begin{aligned} \mathbf{K}_g &= h \cdot \int_A \left(\frac{\partial \delta \mathbf{E}}{\partial u_j} \mathbf{S} \right) dA \\ &= h \cdot \int_A \frac{1}{2} \left(\frac{\partial g_{\alpha}}{\partial u_i} \frac{\partial g_{\beta}}{\partial u_j} + \frac{\partial g_{\alpha}}{\partial u_j} \frac{\partial g_{\beta}}{\partial u_i} \right) S^{\alpha\beta} G_{\alpha} \otimes G_{\beta} \;\; dA \end{aligned} \tag{2-42}$$

this is the geometrical stiffness matrix.

The second term of equation 2-36 is obtained with:

$$\frac{\partial \mathbf{S}}{\partial u_j} = \frac{\partial \mathbf{S}}{\partial \mathbf{E}} \frac{\partial \mathbf{E}}{\partial u_j} = \mathbf{D} : \frac{1}{2}\left[\left(\frac{\partial g_\alpha}{\partial u_i} g_\beta + g_\alpha \frac{\partial g_\beta}{\partial u_i}\right)\right] \tag{2-43}$$

where **D** is a constitutive material tensor.

$$\begin{aligned}\mathbf{K}_m &= h \cdot \int_A \left(\delta \mathbf{E} \frac{\partial \mathbf{S}}{\partial u_j}\right) dA \\ &= h \cdot \int_A \left(\frac{1}{2}\left(\frac{\partial g_\alpha}{\partial u_i} g_\beta + g_\alpha \frac{\partial g_\beta}{\partial u_i}\right)\right) \mathbf{D} : \frac{1}{2}\left[\left(\frac{\partial g_\alpha}{\partial u_i} g_\beta + g_\alpha \frac{\partial g_\beta}{\partial u_i}\right)\right] dA\end{aligned} \tag{2-44}$$

this is the material stiffness matrix.

The total stiffness matrix is given by:

$$\mathbf{K}_T = \mathbf{K}_g + \mathbf{K}_m \tag{2-45}$$

2.4.3 Membrane elements

The membrane elements that will be used in the pneumatic structures examples will be presented in this section. Quadrilateral and triangular membrane elements are implemented to discretize the pneumatic structures.

Shape functions and the derivatives of shape functions w.r.t. to the isoparametric coordinates (ξ^1 and ξ^2) are presented as follows. This equations are applied to calculate the base vectors, stiffness matrix, internal and external forces, displacements, strains, and stresses.

2.4.3.1 Triangular elements

Linear and quadratic elements are shown in Figure 2.4 with 3 and 6 nodes respectively. The number of gauss points used in the numerical integration is also represented in Figure 2.4 with one gauss point for the linear element and 3 gauss points for the quadratic element.

The shape functions for the linear triangular element are given from equation 2-46a to 2-46c.

$$N_1 = 1.0 - \xi^1 - \xi^2 \tag{2-46a}$$

$$N_2 = \xi^1 \tag{2-46b}$$

$$N_3 = \xi^2 \tag{2-46c}$$

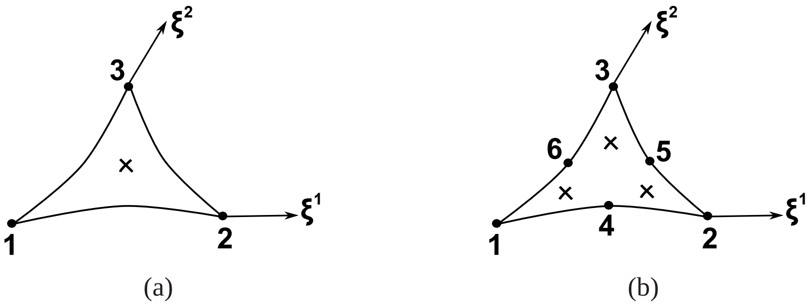

Figure 2.4: Triangular elements: (a) linear and (b) quadratic

The derivatives of the shape functions 2-46a, 2-46b, and 2-46c w.r.t ξ^1 are presented in equation 2-47a to 2-47c and the derivatives of the same shape functions w.r.t. ξ^2 are shown in equation 2-47d to 2-47f.

$$\frac{dN_1}{d\xi^1} = -1.0 \quad (2\text{-}47a) \qquad \frac{dN_1}{d\xi^2} = -1.0 \quad (2\text{-}47d)$$

$$\frac{dN_2}{d\xi^1} = 1.0 \quad (2\text{-}47b) \qquad \frac{dN_2}{d\xi^2} = 0.0 \quad (2\text{-}47e)$$

$$\frac{dN_3}{d\xi^1} = 0.0 \quad (2\text{-}47c) \qquad \frac{dN_3}{d\xi^2} = 1.0 \quad (2\text{-}47f)$$

Equations 2-48a to 2-48f are the shape functions for the quadratic triangular element.

$$N_1 = 2(\xi^1 - 1 + \xi^2)(\xi^1 - \frac{1}{2} + \xi^2) \quad (2\text{-}48a) \qquad N_4 = 4\xi^1(1 - \xi^1 - \xi^2) \quad (2\text{-}48d)$$

$$N_2 = 2\xi^1\xi^1 - \xi^1 \quad (2\text{-}48b) \qquad N_5 = 4\xi^1\xi^2 \quad (2\text{-}48e)$$

$$N_3 = 2\xi^2\xi^2 - \xi^2 \quad (2\text{-}48c) \qquad N_6 = 4\xi^2(1 - \xi^1 - \xi^2) \quad (2\text{-}48f)$$

The derivatives of the shape functions 2-48a to 2-48f w.r.t. ξ^1 are shown in equation 2-49a to 2-49f and the derivatives of the same shape functions w.r.t. ξ^2 are presented in equation 2-49g to 2-49l.

$$\frac{dN_1}{d\xi^1} = 4\xi^1 - 3 + 4\xi^2 \quad (2\text{-}49a) \qquad \frac{dN_1}{d\xi^2} = 4\xi^1 - 3 + 4\xi^2 \quad (2\text{-}49g)$$

$$\frac{dN_2}{d\xi^1} = 4\xi^1 - 1 \quad (2\text{-}49b) \qquad \frac{dN_2}{d\xi^2} = 0 \quad (2\text{-}49h)$$

$$\frac{dN_3}{d\xi^1} = 0 \quad (2\text{-}49c) \qquad \frac{dN_3}{d\xi^2} = 4\xi^2 - 1 \quad (2\text{-}49i)$$

$$\frac{dN_4}{d\xi^1} = 4 - 8\xi^1 - 4\xi^2 \quad (2\text{-}49d) \qquad \frac{dN_4}{d\xi^2} = -4\xi^1 \quad (2\text{-}49j)$$

$$\frac{dN_5}{d\xi^1} = 4\xi^2 \quad (2\text{-}49e) \qquad \frac{dN_5}{d\xi^2} = 4\xi^1 \quad (2\text{-}49k)$$

$$\frac{dN_6}{d\xi^1} = -4\xi^2 \quad (2\text{-}49f) \qquad \frac{dN_6}{d\xi^2} = 4 - 4\xi^1 - 8\xi^2 \quad (2\text{-}49l)$$

2.4.3.2
Quadrilateral elements

Figure 2.5(a) shows the linear quadrilateral element with 4 nodes and full gauss point integration and figure 2.5(b) represents the quadratic quadrilateral element with 9 nodes and reduced gauss point integration.

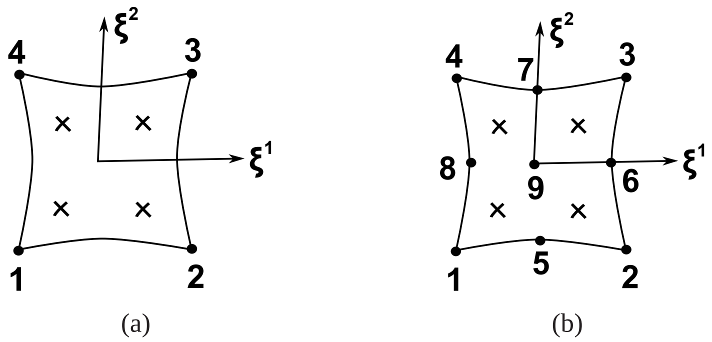

Figure 2.5: Quadrilateral elements: (a) linear and (b) quadratic

From equation 2-50a to 2-50d the shape functions of the linear quadrilateral element are presented.

$$N_1 = \frac{1}{4}(1 - \xi^1)(1 - \xi^2) \quad (2\text{-}50a) \qquad N_3 = \frac{1}{4}(1 + \xi^1)(1 + \xi^2) \quad (2\text{-}50c)$$

$$N_2 = \frac{1}{4}(1 + \xi^1)(1 - \xi^2) \quad (2\text{-}50b) \qquad N_4 = \frac{1}{4}(1 - \xi^1)(1 + \xi^2) \quad (2\text{-}50d)$$

The derivatives of the shape functions of the linear quadrilateral element are given by equation 2-51a to 2-51h.

$$\frac{dN_1}{d\xi^1} = -\frac{1}{4}(1-\xi^2) \quad (2\text{-}51a) \qquad \frac{dN_1}{d\xi^2} = -\frac{1}{4}(1-\xi^1) \quad (2\text{-}51e)$$

$$\frac{dN_2}{d\xi^1} = \frac{1}{4}(1-\xi^2) \quad (2\text{-}51b) \qquad \frac{dN_2}{d\xi^2} = -\frac{1}{4}(1+\xi^1) \quad (2\text{-}51f)$$

$$\frac{dN_3}{d\xi^1} = \frac{1}{4}(1+\xi^2) \quad (2\text{-}51c) \qquad \frac{dN_3}{d\xi^2} = \frac{1}{4}(1+\xi^1) \quad (2\text{-}51g)$$

$$\frac{dN_4}{d\xi^1} = -\frac{1}{4}(1+\xi^2) \quad (2\text{-}51d) \qquad \frac{dN_4}{d\xi^2} = \frac{1}{4}(1-\xi^1) \quad (2\text{-}51h)$$

The shape functions of the quadratic quadrilateral element are presented from equation 2-52a to 2-52i.

$$N_1 = \frac{1}{4}\xi^1\xi^2(\xi^2-1)(\xi^1-1) \quad (2\text{-}52a)$$

$$N_6 = -\frac{1}{2}\xi^1({\xi^2}^2-1)(\xi^1+1) \quad (2\text{-}52f)$$

$$N_2 = \frac{1}{4}\xi^1\xi^2(\xi^2-1)(\xi^1+1) \quad (2\text{-}52b)$$

$$N_7 = -\frac{1}{2}\xi^2({\xi^1}^2-1)(\xi^2+1) \quad (2\text{-}52g)$$

$$N_3 = \frac{1}{4}\xi^1\xi^2(\xi^2+1)(\xi^1+1) \quad (2\text{-}52c)$$

$$N_8 = -\frac{1}{2}\xi^1({\xi^2}^2-1)(\xi^1-1) \quad (2\text{-}52h)$$

$$N_4 = \frac{1}{4}\xi^1\xi^2(\xi^2+1)(\xi^1-1) \quad (2\text{-}52d)$$

$$N_9 = (1-{\xi^1}^2)(1-{\xi^2}^2) \quad (2\text{-}52i)$$

$$N_5 = -\frac{1}{2}\xi^2({\xi^1}^2-1)(\xi^2-1) \quad (2\text{-}52e)$$

The derivatives of the shape functions of the quadratic quadrilateral element w.r.t. ξ^1 are given by equation 2-53a to 2-53i and the derivatives w.r.t. ξ^2 are given by equation 2-54a to 2-54i.

$$\frac{dN_1}{d\xi^1} = \frac{1}{4}\xi^2(\xi^2-1)(2\xi^1-1) \quad (2\text{-}53a)$$

$$\frac{dN_6}{d\xi^1} = -\frac{1}{2}((\xi^2)^2-1)(2\xi^1+1) \quad (2\text{-}53f)$$

$$\frac{dN_2}{d\xi^1} = \frac{1}{4}\xi^2(\xi^2-1)(2\xi^1+1) \quad (2\text{-}53b)$$

$$\frac{dN_7}{d\xi^1} = -\xi^1\xi^2(\xi^2+1) \quad (2\text{-}53g)$$

$$\frac{dN_3}{d\xi^1} = \frac{1}{4}\xi^2(\xi^2+1)(2\xi^1+1) \quad (2\text{-}53c)$$

$$\frac{dN_8}{d\xi^1} = -\frac{1}{2}((\xi^2)^2-1)(2\xi^1-1) \quad (2\text{-}53h)$$

$$\frac{dN_4}{d\xi^1} = \frac{1}{4}\xi^2(\xi^2+1)(2\xi^1-1) \quad (2\text{-}53d)$$

$$\frac{dN_9}{d\xi^1} = (2(\xi^2)^2-2)\xi^1 \quad (2\text{-}53i)$$

$$\frac{dN_5}{d\xi^1} = -\xi^1\xi^2(\xi^2-1) \quad (2\text{-}53e)$$

$$\frac{dN_1}{d\xi^2} = \frac{1}{4}\xi^1(\xi^1 - 1)(2\xi^2 - 1) \quad \text{(2-54a)}$$

$$\frac{dN_2}{d\xi^2} = \frac{1}{4}\xi^1(\xi^1 + 1)(2\xi^2 - 1) \quad \text{(2-54b)}$$

$$\frac{dN_3}{d\xi^2} = \frac{1}{4}\xi^1(\xi^1 + 1)(2\xi^2 + 1) \quad \text{(2-54c)}$$

$$\frac{dN_4}{d\xi^2} = \frac{1}{4}\xi^1(\xi^1 - 1)(2\xi^2 + 1) \quad \text{(2-54d)}$$

$$\frac{dN_5}{d\xi^2} = -\frac{1}{2}((\xi^1)^2 - 1)(2\xi^2 - 1) \quad \text{(2-54e)}$$

$$\frac{dN_6}{d\xi^2} = -\xi^1\xi^2(\xi^1 + 1) \quad \text{(2-54f)}$$

$$\frac{dN_7}{d\xi^2} = -\frac{1}{2}((\xi^1)^2 - 1)(2\xi^2 + 1) \quad \text{(2-54g)}$$

$$\frac{dN_8}{d\xi^2} = -\xi^1\xi^2(\xi^1 - 1) \quad \text{(2-54h)}$$

$$\frac{dN_9}{d\xi^2} = (2(\xi^1)^2 - 2)\xi^2 \quad \text{(2-54i)}$$