5
Pressure-Volume Coupling

One special characteristic of pneumatic structures which distinguishes its mechanical behavior from other membrane structures is the pressure-volume coupling.

According to Jarasjarungkiat [75] numerical examples demonstrate not only the efficiency of the pressure-volume coupling model but also the need to consider the volume (pressure) variation in addition to the change of surface normal vector. The study of Jarasjarungkiat [75] reveals the observable feature that the pressure of an enclosed fluid provides additional stiffness to the inflatable structure, analogous to the behavior of a membrane on elastic springs.

The formulation of the pressure-volume coupling recalls the concept of deformation-dependent forces. The formulation used in the present study refers to the works of Hassler and Schweizerhof [17], Rumpel and Schweizerhof [18], Rumpel [19], Bonet et. al. [20], and Berry and Yang [21].

Hassler and Schweizerhof [17] presented a formulation for the static interaction of fluid and gas for large deformation in finite element analysis that can be applied to pneumatic structures. Moreover it provides a realistic and general description of the interaction of arbitrarily combined fluid and/or gas loaded or filled multi-chamber systems undergoing large deformations.

The use of a deformation-dependent force formulation brings along the drawback of a fully-populated stiffness matrix for which triangular factorization requires large numerical effort. To circumvent this problem Woodbury's formula was used to obtain the inverse of the fully-populated stiffness matrix as discussed in the work of Hager [76]. The Woodbury's formula updates the inverse of a matrix with the update tensors without performing a new factorization of the stiffness matrix.

To validate the pressure-volume coupling formulation, analytical solutions already developed for a circular inflated membrane clamped at its rim is presented. Since the analytical formulation available in the literature ([77] and [78]) is restricted to small strains conditions, an analytical formulation for large strains is developed. The results obtained with analytical solutions are compared with the numerical solutions with and without pressure-volume coupling.

5.1
Numerical analysis model for one chamber

The formulation presented in the work of Hassler and Schweizerhof [17] concern an enclosed volume filled with combined liquid and gas. Rumpel and

Schweizerhof [18] treat the case of structures filled with gas, which is the most common case in civil engineering and will therefore be adopted here.

Taking the principle of virtual work as basis for the problem formulation, the external virtual work of the pressure load is given by:

$$\delta W_{press} = \int_a p\,\mathbf{n} \cdot \delta\mathbf{u}\, da \tag{5-1}$$

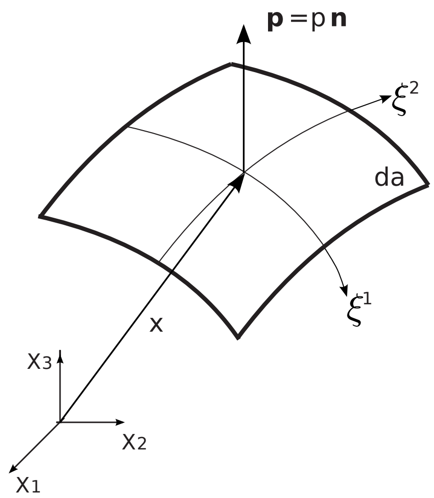


Figure 5.1: Surface under pressure loading.

where $\mathbf{n} = \mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2} / \left|\mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2}\right|$ is the surface normal vector, $da = \left|\mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2}\right| d\xi^1 d\xi^2$ is the surface element, and $p = p(v(\mathbf{x}))$ is the internal pressure. The surface position vector $\mathbf{x}(\xi^1, \xi^2)$ is a function of the local coordinates ξ^1 and ξ^2 represented in figure 5.1. Substituting these definitions in equation 5-1 gives:

$$\begin{aligned}\delta W_{press} &= \int_{\xi^2}\int_{\xi^1} p \frac{\mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2}}{\left|\mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2}\right|} \cdot \delta\mathbf{u} \left|\mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2}\right| d\xi^1 d\xi^2 \\ &= \int_{\xi^2}\int_{\xi^1} p\,(\mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2})\,\delta\mathbf{u} d\xi^1 d\xi^2 = \int_{\xi^2}\int_{\xi^1} p\,\mathbf{n}^* \cdot \delta\mathbf{u}\, d\xi^1 d\xi^2\end{aligned} \tag{5-2}$$

where $\mathbf{n}^* = \mathbf{x}_{,\xi^1} \times \mathbf{x}_{,\xi^2}$.

According to Poisson's law, the constitutive behavior of the gas is described by the following equation:

$$p_i v_i^\kappa = P_i V_i^\kappa = const \tag{5-3}$$

where κ is the isentropy constant, P_i and V_i are the initial pressure and volume and p_i and v_i are the current pressure and volume for each closed chamber i. This equation shows that when the volume decreases (increases) the internal pressure inside the enclosed volume increases (decreases).

When $\kappa = 1$ the adiabatic change simplifies to Boyle-Mariotte's law.

The volume for the enclosed chamber v_i is computed through the equation:

$$v_i = \frac{1}{3}\int_{\xi^2}\int_{\xi^1} \mathbf{x}\cdot\mathbf{n}^*\, d\xi^1 d\xi^2 \tag{5-4}$$

The external virtual work is linearized at state t for the solution with a Newton scheme. Equation 5-2 and the constraint 5-3 are expanded into a Taylor series up to the first order term:

$$\delta W_{i,press}^{lin} = \delta W_{press,t} + \delta W_{press,t}^{\Delta p} + \delta W_{press,t}^{\Delta\mathbf{n}} \tag{5-5}$$

$$\delta W_{i,press}^{lin} = \int_{\xi^2}\int_{\xi^1} \left(p\mathbf{n}^*\cdot\delta\mathbf{u} + \Delta p\mathbf{n}^*\cdot\delta\mathbf{n}^* + p\Delta\mathbf{n}^*\cdot\delta\mathbf{u}\right) d\xi^1 d\xi^2$$

with

$$\Delta\mathbf{n}^* = \Delta\mathbf{u}_{,\xi^1}\times\mathbf{x}_{,\xi^2} - \Delta\mathbf{u}_{,\xi^2}\times\mathbf{x}_{,\xi^1} \tag{5-6}$$

$$\Delta(pv^\kappa) = 0 \tag{5-7}$$

$$\Delta p\cdot v_t^\kappa + \Delta v^\kappa\cdot p_t = 0$$

where

$$\Delta v^\kappa = \kappa\frac{v_t^\kappa}{v_t}\Delta v \tag{5-8}$$

$$\Delta v = \frac{1}{3}\int_{\xi^2}\int_{\xi^1}\left[\Delta\mathbf{u}\cdot\mathbf{n}^* + \mathbf{x}\cdot\Delta\mathbf{n}^*\right] d\xi^1 d\xi^2 = \Delta v^{\Delta\mathbf{u}} + \Delta v^{\Delta\mathbf{n}} \tag{5-9}$$

Equation 5-7 results in:

$$\Delta p + \frac{\kappa p_t}{v_t}\Delta v = 0 \tag{5-10}$$

In the present work the final results for the partial integrations of equation 5-5 will be presented. The solution for each part of the partial integration of the external virtual work are calculated in the works of Hassler and Schweizerhof [17], Rumpel and Schweizerhof [18], and Rumpel [19]. The linearized external virtual work due to the change in the normal vector is given by:

$$\delta W^{\Delta\mathbf{n}}_{press,t} = $$
$$= \frac{p_t}{2}\int_{\xi^2}\int_{\xi^1}\begin{pmatrix}\delta\mathbf{u}\\ \delta\mathbf{u}_{,\xi^1}\\ \delta\mathbf{u}_{,\xi^2}\end{pmatrix}\cdot\begin{bmatrix}0 & \underline{\mathbf{W}}^{\xi^1} & \underline{\mathbf{W}}^{\xi^2}\\ \underline{\mathbf{W}}^{\xi^1 T} & 0 & 0\\ \underline{\mathbf{W}}^{\xi^2 T} & 0 & 0\end{bmatrix}\begin{pmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{u}_{,\xi^1}\\ \Delta\mathbf{u}_{,\xi^2}\end{pmatrix}d\xi^1 d\xi^2 \tag{5-11}$$

where $\underline{\mathbf{W}}^{\xi^1} = \mathbf{n}\otimes\mathbf{x}_{,\xi^1} - \mathbf{x}_{,\xi^1}\otimes\mathbf{n}$ and $\underline{\mathbf{W}}^{\xi^2} = \mathbf{n}\otimes\mathbf{x}_{,\xi^2} - \mathbf{x}_{,\xi^2}\otimes\mathbf{n}$.

The linearized external virtual work due to the change in the pressure is:

$$\delta W^{\Delta p}_{press,t} = -\frac{\kappa p_t}{v_t}\int_{\xi^2}\int_{\xi^1}\mathbf{n}^*\cdot\Delta\mathbf{u}\, d\xi^1 d\xi^2 \int_{\xi^2}\int_{\xi^1}\mathbf{n}^*\cdot\delta\mathbf{u}\, d\xi^1 d\xi^2 \tag{5-12}$$

Replacing equations 5-11 and 5-12 in equation 5-5 gives:

$$\begin{aligned}\delta W^{\Delta p}_{press,t} + \delta W^{\Delta\mathbf{n}}_{press,t} &= -\delta W_{press,t}\\
-\frac{\kappa p_t}{v_t}\int_{\xi^2}\int_{\xi^1}\mathbf{n}^*\cdot\Delta\mathbf{u}\, d\xi^1 d\xi^2 &\int_{\xi^2}\int_{\xi^1}\mathbf{n}^*\cdot\delta\mathbf{u}\, d\xi^1 d\xi^2\\
+\frac{p_t}{2}\int_{\xi^2}\int_{\xi^1}\begin{pmatrix}\delta\mathbf{u}\\ \delta\mathbf{u}_{,\xi^1}\\ \delta\mathbf{u}_{,\xi^2}\end{pmatrix}&\cdot\begin{bmatrix}0 & \underline{\mathbf{W}}^{\xi^1} & \underline{\mathbf{W}}^{\xi^2}\\ \underline{\mathbf{W}}^{\xi^1 T} & 0 & 0\\ \underline{\mathbf{W}}^{\xi^2 T} & 0 & 0\end{bmatrix}\begin{pmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{u}_{,\xi^1}\\ \Delta\mathbf{u}_{,\xi^2}\end{pmatrix}d\xi^1 d\xi^2\\
&= -p_t\int_{\xi^2}\int_{\xi^1}\mathbf{n}^*\cdot\delta\mathbf{u}\, d\xi^1 d\xi^2\end{aligned} \tag{5-13}$$

The discretization for the finite elements is applied taking the equations 5-13 and the isoparametric representation:

$$\mathbf{x} = \mathbf{N}_i\mathbf{x}, \qquad \Delta\mathbf{u} = \mathbf{N}_i\mathbf{d} \quad and \quad \delta\mathbf{u} = \mathbf{N}_i\delta\mathbf{d} \tag{5-14}$$

where $\mathbf{N}_i$ are the shape functions.

The global stiffness matrix and the global load vector are given:

$$\left[\mathbf{K}_T - (\mathbf{K}_{press} - b\mathbf{a}\otimes\mathbf{a})\right]\mathbf{d} = \mathbf{f}_{ext} + \mathbf{f}_{press} - \mathbf{f}_{int} \tag{5-15}$$

$$\mathbf{K}_{press} = $$
$$= \frac{p_t}{2}\int_{\xi^2}\int_{\xi^1}\begin{pmatrix}\delta\mathbf{N}\\ \delta\mathbf{N}_{,\xi^1}\\ \delta\mathbf{N}_{,\xi^2}\end{pmatrix}^T\begin{bmatrix}0 & \underline{\mathbf{W}}^{\xi^1} & \underline{\mathbf{W}}^{\xi^2}\\ \underline{\mathbf{W}}^{\xi^1 T} & 0 & 0\\ \underline{\mathbf{W}}^{\xi^2 T} & 0 & 0\end{bmatrix}\begin{pmatrix}\Delta\mathbf{N}\\ \Delta\mathbf{N}_{,\xi^1}\\ \Delta\mathbf{N}_{,\xi^2}\end{pmatrix}d\xi^1 d\xi^2 \tag{5-16}$$

$$\mathbf{a} = \int_{\xi^2}\int_{\xi^1} \mathbf{N}^T\mathbf{n}^*\, d\xi^1 d\xi^2 \tag{5-17}$$

$$\mathbf{f}_{press} = p_t \int_{\xi^2}\int_{\xi^1} \mathbf{N}^T\mathbf{n}^*\, d\xi^1 d\xi^2 \tag{5-18}$$

$$b = \kappa \frac{p_t}{v_t} \tag{5-19}$$

where $\mathbf{K}_T$ is the total stiffness matrix containing the geometrical and material stiffness, $\mathbf{K}_{press}$ is the load stiffness matrix for each structural element in contact with gas, $\mathbf{a}$ is the coupling vector, $\mathbf{f}_{press}$ is the load vector, $\mathbf{f}_{int}$ is the force residuum vector, and $\mathbf{f}_{ext}$ is the vector of the external forces. According to Rumpel [19] the symmetric load stiffness matrix $\mathbf{K}_{press}$ reflects the effect of the direction–dependent internal pressure and the fully–populated coupling matrix $b\mathbf{a}\otimes\mathbf{a}$ is the volume–dependent internal pressure contribution.

Equation 5-15 can be rewritten as:

$$[\mathbf{K}^* + b\mathbf{a}\otimes\mathbf{a}]\,\mathbf{d} = \mathbf{F} \tag{5-20}$$

where $\mathbf{K}^* = \mathbf{K}_T - \mathbf{K}_{press}$ and $\mathbf{F} = \mathbf{f}_{ext} + \mathbf{f}_{press} - \mathbf{f}_{int}$.

The stiffness matrix is fully-populated, and therefore triangular factorization requires great computational effort. To circumvent this problem the Sherman-Morrison-Woodbury formula is used to solve the fully-populated stiffness matrix, as discussed in the work of Hager [76].

5.1.1 Sherman-Morrison-Woodbury formula

As presented by Hager [76] this formula relates the inverse of a matrix after a small rank perturbation to the inverse of the original matrix dismissing factorization. The focus is on the following result. If both $\mathbf{A}$ and $\mathbf{I} - \mathbf{V}\mathbf{A}^{-1}\mathbf{U}$ are invertible, then $\mathbf{A} - \mathbf{U}\mathbf{V}$ is invertible and:

$$[\mathbf{A} - \mathbf{U}\mathbf{V}]^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{U}(\mathbf{I} - \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1} \tag{5-21}$$

where $\mathbf{U}\mathbf{V}$ is given by equation 5-22 supposing that $\mathbf{U}$ is $n \times m$ with columns $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_m$ and $\mathbf{V}$ is $m \times n$ with rows $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$

$$\mathbf{U}\mathbf{V} = \sum_{i=1}^{m} \mathbf{u}_i\mathbf{v}_i \tag{5-22}$$

In the special case where **U** is a column vector **u** and **V** is a row vector **u**, equation 5-21 simplifies to:

$$[\mathbf{A} - \mathbf{uv}]^{-1} = \mathbf{A}^{-1} + \alpha\mathbf{A}^{-1}\mathbf{uvA}^{-1} \tag{5-23}$$

where $\alpha = 1/(1 - \mathbf{vA}^{-1}\mathbf{u})$

To solve the linear system $\mathbf{Bx} = \mathbf{b}$ where $\mathbf{B} = \mathbf{A} - \mathbf{UV}$ equation 5-21 is used to calculate the inverse of **B**:

$$\begin{aligned}
\mathbf{x} &= \mathbf{B}^{-1}\mathbf{b} \\
\mathbf{x} &= \left[\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{U}(\mathbf{I} - \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1}\right]\mathbf{b} \\
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} + \mathbf{A}^{-1}\mathbf{U}(\mathbf{I} - \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1}\mathbf{b} \\
\mathbf{x} &= \mathbf{y} + \mathbf{WC}^{-1}\mathbf{Vy} \\
\mathbf{x} &= \mathbf{y} + \mathbf{Wz}
\end{aligned} \tag{5-24}$$

If **V** is $m\times n$, where m is much smaller than n, then the rank of the modification **UV** is small relative to the dimension n of **A** and the system of m linear equations $\mathbf{z} = \mathbf{C}^{-1}\mathbf{Vy}$ is solved quickly. If $m = 1$ then **z** is a scalar **Vy/C**. This is the case of a pneumatic structure with one chamber.

5.2
Multichambers structures

According to Hassler and Schweizerhof [17] the procedure for single chamber membrane can be directly expanded to multiple gas filled chambers connected to each other. Stiffness matrices, coupling vectors and right-hand side vectors in equation 5-15 depicted by index i have to be set up for each chamber i and must be summed up for all n chambers:

$$\left[\mathbf{K}_T - \sum_{i=1}^{n}\left[\mathbf{K}_{press_i} + b\mathbf{a}_i \otimes \mathbf{a}_i\right]\right]\mathbf{d} = \mathbf{f}_{ext} - \mathbf{f}_{int} + \sum_{i=1}^{n}\left[\mathbf{f}_{press_i}\right] \tag{5-25}$$

5.3
Analytical solution for a circular inflated membrane

A circular inflated membrane clamped at its rim is inflated by a uniform pressure. The membrane is supposed to have large displacements. An analytical formulation was proposed by Hencky (apud Fichter [77]), Fichter [77], and Campbell [78] for membrane under small strain conditions. Fichter [77] considered that the pressure remains orthogonal to the membrane during the inflation. One the other hand, by Hencky the pressure remains vertical to the z–axis (see figure 5.2) during the inflation. Fichter shows that this consideration results in an additive term in the equation of the radial equilibrium. This additional term will be show as follows. Campbell [78] generalized Hencky's problem to include the influence of an arbitrary initial tension.

In the present work an analytical solution is developed for inflated circular membranes considering that the pressure remains orthogonal to the surface during the inflation and an arbitrary initial tension in the membrane. The effects of large strains are incorporated in the new analytical solution.

5.3.1 Hencky's solution

Hencky's solution considers a uniform lateral loading, i.e. , the radial component of pressure on the deformed membrane is neglected. The equation for radial equilibrium is:

$$N_\theta = \frac{d}{dr}(r \cdot N_r) \tag{5-26}$$

and for circumferential equilibrium:

$$N_r \frac{d}{dr}(w) = -\frac{pr}{2} \tag{5-27}$$

r and θ are the radial and circumferential coordinates respectively and N_r and N_θ are the corresponding stress resultants, w is the vertical deflection, and p is the uniform lateral loading. Figure 5.2 shows the radial and circumferential coordinates, vertical deflection, and radial displacement of the circular membrane.

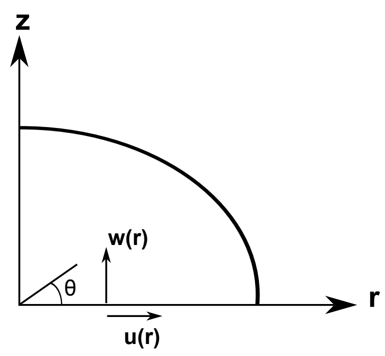


Figure 5.2: Radial and circumferential coordinates, vertical deflection, and radial displacement of a circular membrane

Linear elastic behavior is assumed for the material, thus the stress-strain relations are:

$$N_\theta - \mu \cdot N_r = E \cdot h \cdot \epsilon_\theta \tag{5-28}$$

$$N_r - \mu \cdot N_\theta = E \cdot h \cdot \epsilon_r \tag{5-29}$$

where h is the thickness of membrane.

The strain-displacement relation is given by:

$$\epsilon_r = \frac{d}{dr}(u) + \frac{1}{2} \cdot \left(\frac{dw}{dr}\right)^2 \tag{5-30}$$

$$\epsilon_\theta = \frac{u}{r} \tag{5-31}$$

where u is the radial displacement and μ is the Poisson ratio.

The boundary conditions are:

$$w(a) = 0 \tag{5-32}$$

$$u(a) = 0 \tag{5-33}$$

where a is the membrane radius.

Combining equations 5-26 through 5-31, and defining dimensionless quantities $W = w/a$, $N = N_r/(Eh)$, $\rho = r/a$ and $q = pa/(Eh)$, the resulting equations are:

$$\rho \frac{d}{d\rho}\left[\frac{d}{d\rho}(\rho N) + N\right] + \frac{1}{2}\left(\frac{dW}{d\rho}\right)^2 = 0 \tag{5-34}$$

$$N\frac{dW}{d\rho} = -\frac{1}{2}q\rho \tag{5-35}$$

Substitution of equation 5-35 into equation 5-34 gives:

$$N^2 \frac{d}{d\rho}\left[\frac{d}{d\rho}(\rho N) + N\right] + \frac{1}{8}q^2\rho = 0 \tag{5-36}$$

Hencky considered the solution for stress resultant $N(\rho)$ in the form of a power series:

$$N(\rho) = \frac{1}{4}q^{2/3}\sum_0^\infty b_{2n}\rho^{2n} \tag{5-37}$$

Substitution of $N(\rho)$ in equation 5-36 gives:

$$(b_0 + b_2\rho^2 + b_4\rho^4 + b_6\rho^6 + ...)^2(8b_2\rho + 24b_4\rho^3 + 48b_6\rho^5 + 80b_8\rho^7 + ...) = -8\rho \tag{5-38}$$

Matching the coefficients of equation 5-38, yields the relations between b_0, b_2, b_4, ... :

$$b_0^2 b_2 = -1 \tag{5-39}$$

$$2\, b_0 b_2{}^2 + 3\, b_0{}^2 b_4 = 0 \tag{5-40}$$

...

These equations can be solved successively for b_2, b_4, b_6 ... in terms of b_0:

$$b_2 = -\frac{1}{b_0^2}\ ,\ b_4 = -\frac{2}{3b_0^5}\ ,\ b_6 = -\frac{13}{18b_0^8}\ ,\ b_8 = -\frac{17}{18b_0^{11}}\ ,\ b_{10} = -\frac{37}{27b_0^{14}},$$
$$b_{12} = -\frac{1205}{567b_0^{17}}\ ,\ b_{14} = -\frac{219241}{63504b_0^{20}}\ ,\ b_{16} = -\frac{6634069}{1143072b_0^{23}}, \qquad (5\text{-}41)$$
$$b_{18} = -\frac{51523763}{5143824b_0^{26}}\ ,\ b_{20} = -\frac{998796305}{56582064b_0^{-29}}$$

The coefficient b_0 is obtained by imposing the remaining boundary conditions, equation 5-33, and combining equations 5-26, 5-28 and 5-31:

$$\left(\rho\left[\frac{d}{d\rho}(\rho N) + N\right] = 0\right)_{\rho=1} \qquad (5\text{-}42)$$

Application of equation 5-37 gives:

$$\begin{aligned}(-1+\mu)\,b_0 + (-3+\mu)\,b_2 + (-5+\mu)\,b_4 + (-7+\mu)\,b_6 + (-9+\mu)\,b_8 \\ +(-11+\mu)\,b_{10} + (-13+\mu)\,b_{12} + (-15+\mu)\,b_{14} + (-17+\mu)\,b_{16} \\ +(-19+\mu)\,b_{18} + (-21+\mu)\,b_{20} = 0\end{aligned} \qquad (5\text{-}43)$$

Substituting equation 5-41 in equation 5-43, yields the following equation in b_0:

$$\begin{aligned}(-1+\mu)b_0 - \frac{1}{b_0^2}(-3+\mu) - \frac{2}{3b_0^5}(-5+\mu) - \frac{13}{18b_0^8}(-7+\mu) \\ -\frac{17}{18{b_0}^{11}}(-9+\mu) - \frac{37}{27{b_0}^{14}}(-11+\mu) - \frac{1205}{567{b_0}^{17}}(-13+\mu) \\ -\frac{219241}{63504{b_0}^{20}}(-15+\mu) - \frac{6634069}{1143072{b_0}^{23}}(-17+\mu) \\ -\frac{51523763}{5143824{b_0}^{26}}(-19+\mu) - \frac{998796305}{56582064{b_0}^{29}}(-21+\mu) = 0\end{aligned} \qquad (5\text{-}44)$$

The value of b_0 can now be solved for a specified value of μ.

The displacement $W(\rho)$ is also assumed to be in the form of power series:

$$W(\rho) = q^{1/3}\sum_{0}^{\infty} a_{2n}(1-\rho^{2n+2}) \qquad (5\text{-}45)$$

To obtain the coefficients in the series for $W(\rho)$, expressions 5-37 and 5-45 are inserted into equation 5-35:

$$(b_0+b_2\rho^2+b_4\rho^4+b_6\rho^6+b_8\rho^8+...)(a_0+2a_2\rho^2+3a_4\rho^4+4a_6\rho^6+5a_8\rho^8+...) = 1 \quad (5\text{-}46)$$

Equating coefficients in equation 5-46 yields the relations:

$$b_0 a_0 = 1 \tag{5-47}$$

$$2\, b_0 a_2 + b_2 a_0 = 0 \tag{5-48}$$

...

Combination with values for b_n in equation 5-41 gives:

$$\begin{gathered} a_0 = \frac{1}{b_0}\ ,\ a_2 = \frac{1}{2{b_0}^4}\ ,\ a_4 = \frac{5}{9{b_0}^7}\ ,\ a_6 = \frac{55}{72{b_0}^{10}}\ ,\ a_8 = \frac{7}{6{b_0}^{13}} \\ a_{10} = \frac{205}{108\,{b_0}^{16}}\ ,\ a_{12} = \frac{17051}{5292{b_0}^{19}}\ ,\ a_{14} = \frac{2864485}{508032{b_0}^{22}} \\ a_{16} = \frac{103863265}{102876 48{b_0}^{25}}\ ,\ a_{18} = \frac{27047983}{1469664 b_0^{28}}\ ,\ a_{20} = \frac{42367613873}{1244805408{b_0}^{31}} \end{gathered} \tag{5-49}$$

The solution of equation 5-44 gives the value for b_0 and the coefficients in 5-41 and 5-49 are also solved. Substitution of these coefficients into equations 5-37 and 5-45 gives the dimensionless stress resultant $N(\rho)$ and lateral displacement $W(\rho)$.

5.3.2 Fichter's solution

The equation of the radial equilibrium for Fichter's solution has in comparison with Hencky's solution (see equation 5-26), an addition term:

$$N_\theta = \frac{d}{dr}(r \cdot N_r) - p \cdot r\frac{d}{dr}(w) \tag{5-50}$$

This additional term is the normal pressure which is neglected in Hencky's solution. By Fichter's solution the lateral equilibrium and the stress-strain relation remain the same as those of Hencky's solution (see equations 5-27 through 5-31).

The calculation for Fichter's solution is analogous to Hencky's solution, with equations 5-27 through 5-31 and 5-50, and defining dimensionless quantities $W = w/a$, $N = N_r/(pa)$, $\rho = r/a$ and $q = pa/(Eh)$, the resulting equations are:

$$N^2\rho^2\frac{d^2}{d\rho^2}N + \left(3\,N^2\rho - \frac{1}{2}\rho^3\right)\frac{d}{d\rho}N + \alpha\rho^2 N + \frac{1}{8}\,\frac{\rho^2}{q} = 0 \tag{5-51}$$

$$N\frac{dW}{d\rho} = -\frac{1}{2}\rho \tag{5-52}$$

where $\alpha = (3 + \mu)/2$.

The solution for $N(\rho)$ is obtained through a power series:

$$N(\rho) = \sum_{0}^{\infty} n_{2m}\rho^{2m} \tag{5-53}$$

Substituting equation 5-53 into equation 5-51 and equating coefficients n_2, n_4, n_6, n_8, ... This coefficients can be solved in terms of n_0:

$$n_2 = -\frac{1 + 8\,\alpha\, qn0}{64 qn0^2} \tag{5-54}$$

$$n_4 = -\frac{(1 + 8\,\alpha\, qn0)\,(4\, n0\, q + 1 + 4\,\alpha\, qn0)}{6144 n0^5 q^2} \tag{5-55}$$

$$n_6 = -\frac{(1 + 8\,\alpha\, qn0)}{4718592 q^3 n0^8} \cdot \tag{5-56}$$

$$\frac{\left(13 + 128\,\alpha\, qn0 + 256\,\alpha^2 q^2 n0^2 + 128\, n0^2 q^2 + 96\, n0\, q + 576\, n0^2 q^2 \alpha\right)}{4718592 q^3 n0^8}$$

...

The solution in a power series for $W(\rho)$ is given by:

$$W(\rho) = \sum_{0}^{\infty} w_{2n}(1 - \rho^{2n+2}) \tag{5-57}$$

Substituting the power series 5-57 and 5-53 in equation 5-52 and equating the coefficients of powers of ρ gives a system of simultaneous equations and the coefficients w_0, w_2, w_4, w_6, ... result in terms of n_0:

$$w_0 = 1/4\, n0^{-1} \tag{5-58}$$

$$w_2 = \frac{1}{512}\,\frac{1 + 8\,\alpha\, qn0}{qn0^4} \tag{5-59}$$

$$w_4 = \frac{1}{147456}\,\frac{(1 + 8\,\alpha\, qn0)\,(8\, n0\, q + 5 + 32\,\alpha\, qn0)}{n0^7 q^2} \tag{5-60}$$

...

Substituting equations 5-50 and 5-31 into equation 5-28 and applying the boundary conditions for the radial displacement (5-33), gives:

$$r\left(\frac{d}{dr}(rN_r) - \mu N_r - pr\frac{dw}{dr}\right) = u \tag{5-61}$$

$$\left[r\left(\frac{d}{dr}N_r - \mu N_r - pr\frac{dw}{dr}\right]\right)_{r=a} = 0$$

The dimensionless form of equation 5-61 is given by:

$$\left[\rho\left(\frac{d}{d\rho}(\rho N) - \mu N - \rho\frac{dW}{d\rho}\right)\right]_{\rho=1} = 0 \tag{5-62}$$

By specifying values for μ and q, and substituting equations 5-57 and 5-53 in equation 5-62 the value of n_0 is obtained. The value n_0 is used in the explicit truncated series for $N(\rho)$ and $W(\rho)$, which are respectively the dimensionless stress resultant and lateral displacement.

5.3.3
Campbell's solution

Campbell's solution is an extension of Hencky's solution to include the case of an arbitrary pretension (N_0). Therefore the change in the equation 5-27 for the lateral equilibrium considering pretension is:

$$(N_0 + N_r)\frac{d}{dr}(w) = -\frac{pr}{2} \tag{5-63}$$

The radial equilibrium equation, stress-strain relation, and strain-displacement relation, remain the same as those of Hencky's solution.

Therefore Campbell's solution is obtained analogously to Hencky's solution. With equations 5-63, 5-26, 5-28 through 5-31, and defining the dimensionless quantities $W = w/a$, $N = N_r/(Eh)$, $N0 = N_0/(Eh)$, $N\theta = N_\theta/(Eh)$, $\rho = r/a$, and $q = pa/(Eh)$, the resulting equations are:

$$\frac{1}{\rho\, q^2}\,(N + N0)^2\,\frac{dl}{d\rho}\,(N\theta + N) = -\frac{1}{8} \tag{5-64}$$

$$(N + N0)\,\frac{dW}{d\rho} = -\frac{1}{2}q\rho \tag{5-65}$$

The solution for $N(\rho)$ is similar to Hencky's solution (see equation 5-37):

$$N(\rho) = \frac{1}{4}q^{2/3}\sum_{0}^{\infty} b_{2n}\rho^{2n} - N0 \tag{5-66}$$

Substituting $N(\rho)$ in the modified equation for $N\theta(\rho)$ (see equation 5-26) gives

$N\theta(\rho)$:

$$N\theta(\rho) = \frac{d}{d\rho}(\rho N)$$

$$N\theta(\rho) = \rho\frac{d}{d\rho}(N) + N$$

$$\begin{aligned} N\theta(\rho) = \rho\frac{1}{4}q^{2/3}(2b_2\rho + 4b_4\rho_3 + 6b_6\rho_5 + ...)+ \\ \frac{1}{4}q^{2/3}(b_0 + b_2\rho_2 + b_4\rho_4 + ...) - N0 \end{aligned} \tag{5-67}$$

The coefficients b_2, b_4, b_6, b_8, ..., can be solved in terms of b_0, substituting equations 5-66 and 5-67 in equation 5-65. The values of the coefficients b_n are the same as the coefficients of Hencky's solution, which were given in equation 5-41.

The coefficient b_0 is evaluated with equation 5-42 substituting $N(\rho)$, given in equation 5-66, and the coefficients b_n presented in equation 5-41:

$$\begin{aligned}
113164128\,q^{2/3}b0^{30} - 339492384\,q^{2/3}b0^{27} - 377213760\,q^{2/3}b0^{24} \\
-572107536\,q^{2/3}b0^{21} - 961895088\,q^{2/3}b0^{18} - 1705844448\,q^{2/3}b0^{15} \\
-3126483360\,q^{2/3}b0^{12} - 5860311930\,q^{2/3}b0^{9} - 11165138127\,q^{2/3}b0^{6} \\
-21536932934\,q^{2/3}b0^{3} - 41949444810\,q^{2/3} - 452656512\,N0\,b0^{29} \\
-113164128\,\mu\,q^{2/3}b0^{30} + 113164128\,\mu\,q^{2/3}b0^{27} + 75442752\,\mu\,q^{2/3}b0^{24} \\
+81729648\,\mu\,q^{2/3}b0^{21} + 106877232\,\mu\,q^{2/3}b0^{18} + 155076768\,\mu\,q^{2/3}b0^{15} \\
+240498720\,\mu\,q^{2/3}b0^{12} + 390687462\,\mu\,q^{2/3}b0^{9} + 656772831\,\mu\,q^{2/3}b0^{6} \\
+1133522786\,\mu\,q^{2/3}b0^{3} + 1997592610\,\mu\,q^{2/3} + 452656512\,\mu\,N0\,b0^{29} = 0
\end{aligned} \tag{5-68}$$

The value of b_0 can now be solved for specified values of μ, q, and $N0$ with equation 5-68.

The solution for $W(\rho)$ is the same as the one obtained by Hencky's solution (equation 5-45). The coefficients a_n in the power series equation $W(\rho)$, are solved with equations 5-64 and 5-45 in equation 5-65.

With the coefficients a_n and b_n, the explicit truncated series for $N(\rho)$ and $W(\rho)$ are calculated.

5.3.4
Modified Fichter's solution

Initial tension or pretension is applied in most cases of membrane structures. Therefore, an analytical solution with Fichter's solution considering an initial tension is developed in the present work.

The equation of radial equilibrium is the equation of Fichter's solution (equation 5-50) and the equation of lateral equilibrium is the one of Campbell's solution (equation 5-63). These equations are rewritten:

$$N_\theta = \frac{d}{dr}(r \cdot N_r) - p \cdot r\frac{d}{dr}(w)$$

$$(N_0 + N_r)\frac{d}{dr}(w) = -\frac{pr}{2}$$

In this case the solution is analogous to Fichter's solution, with equations 5-63, 5-50, 5-28 through 5-31, and defining dimensionless quantities $W = w/a$, $N = N_r/(pa)$, $N0 = N_0/(pa)$, $\rho = r/a$ and $q = pa/(Eh)$, the resulting equations are:

$$(NR + N0)^2\left(\rho^2\frac{d^2}{d\rho^2}(N) + 3\rho\frac{d}{d\rho}(N)\right) + \alpha\rho^2\,(N + N0) + \\ + \frac{\rho^2}{8q} - \frac{\rho^3}{2}\frac{d}{d\rho}\,(N + N0) = 0 \tag{5-69}$$

$$(N + N0)\,\frac{dW}{d\rho} = -\frac{1}{2}q\rho \tag{5-70}$$

The solution for $N(\rho)$ is similar to Fichter's solution (see equation 5-53):

$$N(\rho) = \sum_0^\infty n_{2m}\rho^{2m} - N0 \tag{5-71}$$

Substituting $N(\rho)$ in equation 5-69 the coefficients n_m are solved in terms of n_0:

$$n_2 = -\frac{1}{64}\cdot\frac{(1 + 8\alpha q n_0)}{q n_0^2} \tag{5-72}$$

$$n_4 = -\frac{1}{6144}\cdot\frac{(1 + 8\alpha q n_0)\,(1 + 4\alpha q n_0 + 4q n_0)}{q^2 n_0^5} \tag{5-73}$$

$$n_6 = -\frac{(1 + 8\alpha\, q n_0)}{4718592}\cdot \\ \frac{\left(13 + 128\alpha\, q n_0 + 96 q n_0 + 256\alpha^2 q^2 {n_0}^2 + 576\alpha q^2 {n_0}^2 + 128\, q^2 {n_0}^2\right)}{q^3 {n_0}^8} \tag{5-74}$$

...

The solution of $W(\rho)$ is the same solution in power series used in Fichter's solution (see equation 5-57). Substituting equations 5-71 and 5-57 into equation 5-70 gives the coefficients w_m.

$$w_0 = \frac{1}{4}n_0^{-1} \tag{5-75}$$

$$w_2 = \frac{1}{512}\frac{1 + 8\alpha q n_0}{q {n_0}^4} \tag{5-76}$$

$$w_4 = \frac{1}{147456}\frac{5 + 72\alpha q n_0 + 8 q n_0 + 256\alpha^2 q^2 {n_0}^2 + 64\alpha q^2 {n_0}^2}{q^2 {n_0}^7} \tag{5-77}$$

...

The explicit truncated series $N(\rho)$ and $W(\rho)$ are calculated with the coefficients n_m and w_m

5.3.5
Finite strain solution

The finite strain solution is obtained through Fichter's solution (see section 5.3.2) and the consideration of finite strain term ($\frac{1}{2} \cdot \left(\frac{du}{dr}\right)^2$) in ϵ_r. The finite strain term ($\frac{1}{2} \cdot \left(\frac{u}{r}\right)^2$) in ϵ_θ is not considered.

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2} \cdot \left(\frac{dw}{dr}\right)^2 + \frac{1}{2} \cdot \left(\frac{du}{dr}\right)^2 \tag{5-78}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{2} \cdot \left(\frac{u}{r}\right)^2 \tag{5-79}$$

where the terms $\frac{du}{dr}$ and $\frac{u}{r}$ account for small strains, the term $\frac{1}{2} \cdot \left(\frac{dw}{dr}\right)^2$ arises in the presence of large displacements and the terms $\frac{1}{2} \cdot \left(\frac{du}{dr}\right)^2$ and $\frac{1}{2} \cdot \left(\frac{u}{r}\right)^2$ account for finite strains.

The calculation for this solution is analogous to the previous solutions, with equations 5-27, 5-50, 5-28 through 5-30, and 5-79, and defining the dimensionless quantities $W = w/a$, $N = N_r/(pa)$, $\rho = r/a$, and $q = pa/(Eh)$, the resulting equations are:

$$\frac{p}{q}\left(A + \frac{1}{2}A^2\right) + \rho\,\mu\,p\frac{d}{d\rho}(N) + N\,p\,(\mu - 1) + \frac{1}{2}\frac{\mu\,p\rho^2}{N} + \frac{1}{8}\frac{p\rho^2}{qN^2} = 0 \tag{5-80}$$

$$A = q\rho^2\frac{d^2}{d\rho^2}(N) + q\rho\frac{d}{d\rho}(N)\left(3 - \mu - \frac{1}{2}\frac{\rho^2}{N^2}\right) + N\,q\,(1-\mu) + \frac{3}{2}\frac{q\rho^2}{N} \tag{5-81}$$

The solution of $N(\rho)$ is the same of Fichter's solution (see equation 5-53). Substituting equation 5-53 in equation 5-80 and equating coefficients n_2, n_4, n_6, n_8, ... This coefficients are solved in terms of n_0:

$$n_2 = \frac{1}{8}\frac{-12\,q^2{n_0}^2 - 12\,qn_0 - 1 + 12\,q^2\mu\,{n_0}^2 - 4\,\mu\,qn_0}{{n_0}^2 q\,(5\,q\mu^2 n_0 + 11\,qn_0 + 8 - 16\,\mu\,qn_0)} \tag{5-82}$$

$$n_4 = -\frac{C}{8\,B} \tag{5-83}$$

$$B = qn_0^{\,3}\left(7\,q\mu^2 n_0 - 34\,\mu\,qn_0 + 27\,qn_0 + 24\right) \tag{5-84}$$

$$\begin{aligned} C = {} & -648\,q^2\mu\,n_2^{\,2}n_0^{\,3} + 636\,q^2 n_0^{\,3}n_2^{\,2} + 12\,\mu\,qn_0\,n_2 + 136\,q^2 n_0^{\,2}n_2 \\ & +256\,qn_2^{\,2}n_0^{\,2} + 28\,qn_2\,n_0 - 64\,q^2\mu\,n_0^{\,2}n_2 + 156\,q^2\mu^2 n_0^{\,3}n_2^{\,2} \\ & +9\,q^2 n_0 + 2\,n_2 \end{aligned} \tag{5-85}$$

...

Substituting equations 5-53 and 5-57 into equation 5-52, gives the coefficients w_0, w_2, w_4, ..., of $W(\rho)$:

$$w_0 = \frac{1}{(4n_0)} \tag{5-86}$$

$$w_2 = -\frac{1}{2}\frac{n_2\,w_0}{n_0} \tag{5-87}$$

$$w_4 = -\frac{1}{6}\frac{2\,n_4\,w_0 + 4\,n_2\,w_2}{n_0} \tag{5-88}$$

...

5.4 Comparison of analytical and numerical analysis

The response of a circular membrane clamped at its rim and inflated by a uniform pressure is analyzed. Solutions for both small and large strain conditions obtained with analytical and numerical models presented in this work are compared.

The data used for the numerical and analytical analysis is from the study of Bouzidi et. al. [79]. The membrane characteristics are: $E = 311488Pa$ (Young's modulus), $\nu = 0.34$ (Poisson ratio) and the radius is 0.1425 m. The static analysis is carried out in two steps. First the configuration for an internal pressure of 400kPa is obtained. After the inflation, external pressures are applied. Bouzidi et. al. [79] consider the circular membrane initially flat and the inflation for pressures of 100kPa, 250kPa and 400kPa are applied. The mesh for the numerical solution is composed of 640 membrane elements (see figure 5.3).

A comparison between a mesh composed by linear and quadratic elements is performed and it is presented in figure 5.4. The linear triangular element (T3) has 3 nodes and 1 gauss integration point and the linear quadrilateral element (Q4) has 4 nodes and 2x2 gauss integration. The quadratic triangular element (T6) has 6

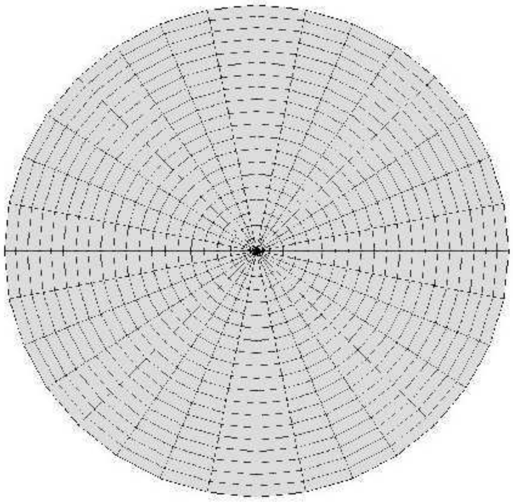


Figure 5.3: Mesh for a circular inflated membrane.

nodes and 3 gauss integration points and the quadratic quadrilateral element (Q9) has 9 nodes and reduced 2x2 gauss integration. The mesh with linear elements has 641 nodes and the mesh with quadratic elements has 2529 nodes. The results of the comparison are the same for the mesh with linear and quadratic elements. Therefore, the mesh with linear elements is chosen in these analysis because of the faster performance.

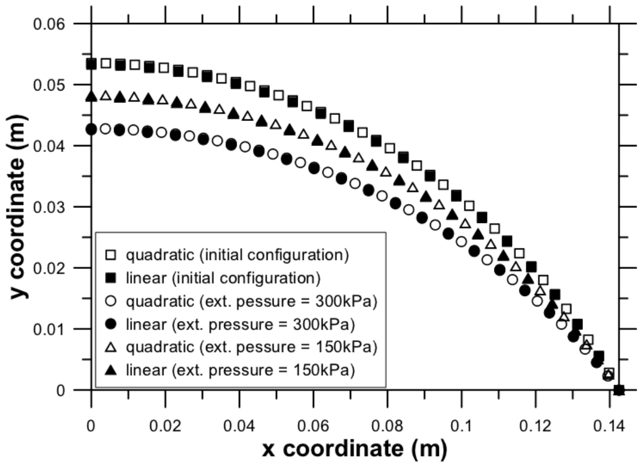


Figure 5.4: Comparison between a mesh with linear and quadratic elements for applied external pressure values of 150kPa and 300kPa.

5.4.1 Results

Figure 5.5 shows the results of Hencky's and Fichter's solutions for the applied external pressures of 150kPa and 300kPa. The difference between both

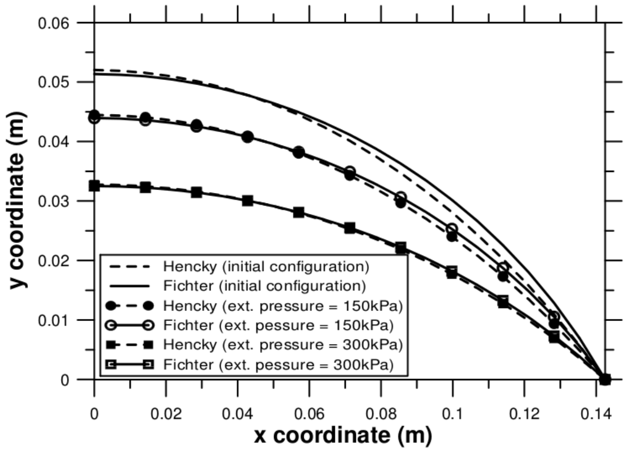


Figure 5.5: Comparison between Hencky's and Fichter's solution for applied external pressure values of 150kPa and 300kPa.

solutions is due to the additional term associated to the normal pressure present only in Fichter's solution.

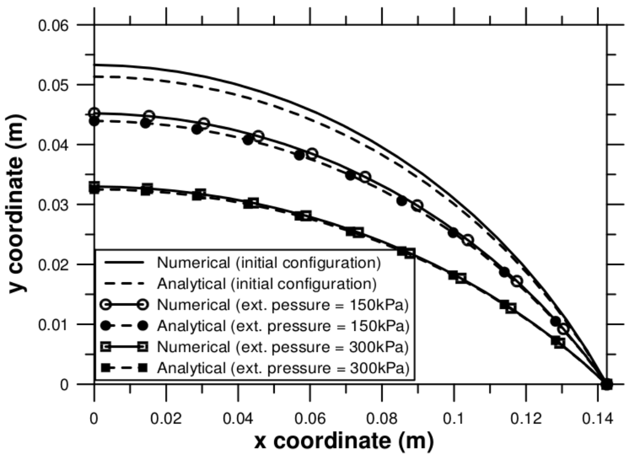


Figure 5.6: Fichter's solution and numerical results without pretension and $\kappa = 0$ for applied external pressures values of 150kPa and 300kPa.

A comparison of Fichter's solution with the numerical those of FEM is presented in figure 5.6. The difference in the result obtained with Fichter's solution and the numerical solution is accredited to the presence of finite strains, which are included in the finite element formulation and are precluded in the analytical solution.

Figure 5.7 presents the results of a numerical solution for the circular membrane with pressure-volume coupling ($\kappa = 1$) and without ($\kappa = 0$). Pressure-volume coupling is more noticeable for higher external pressure values, in agreement with Poisson's law (see equation 5-3). It is important to observe that according to the amount of coupling different final configurations are obtained.

Next, the influence of pretension is investigated. Figure 5.8(a) presents the

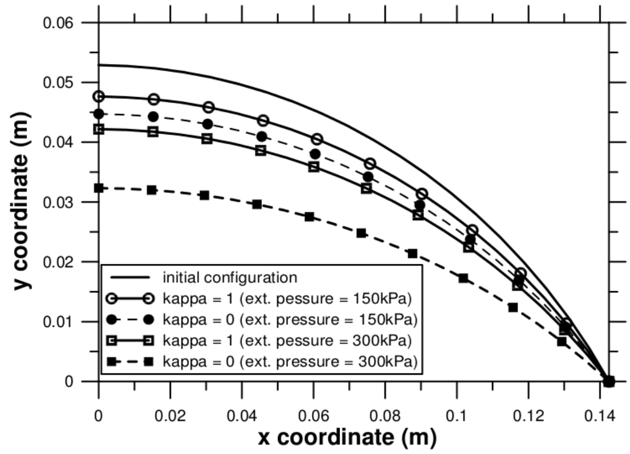
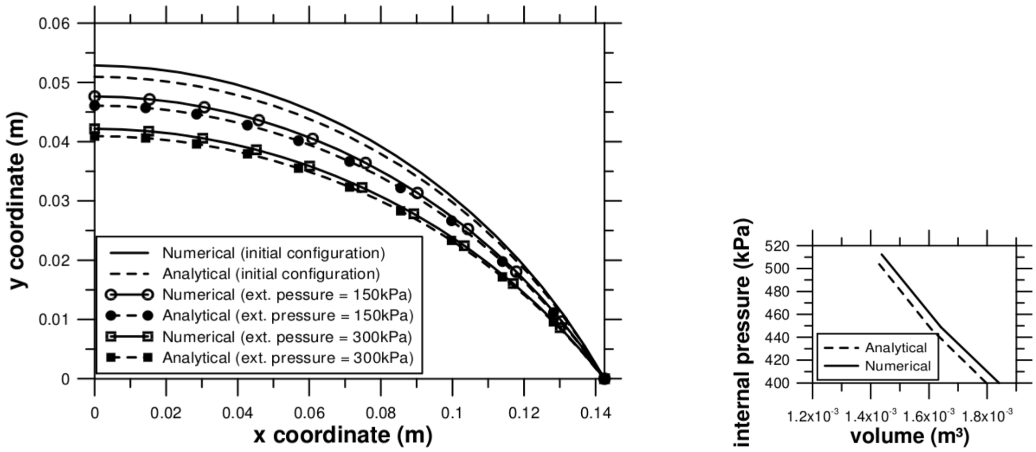


Figure 5.7: Comparison between the numerical solution with a pretension of 1kPa for $\kappa = 0$ and $\kappa = 1$ for applied external pressure values of 150kPa and 300kPa.



(a) (b)

Figure 5.8: Analytical and numerical solution with a pretension of 1kPa and $\kappa = 1$ for an applied external pressure values of 150kPa and 300kPa: (a) deformed configuration and (b) pressure volume curve.

results for the analytical and numerical solution with a pretension of 1kPa and with pressure-volume coupling subjected to external pressures of 150kPa and 300kPa. The analytical solution takes into account both the term from the normal pressure, which is neglected in Hencky's solution, and a pretension on the membrane, which is considered neither in Hencky's nor Fichter's solution. The results obtained with the analytical solution are in accordance with the numerical results. The relation between the internal pressure versus volume are illustrated in figure 5.8(b), stressing that when the volume decreases due to the external pressure the internal pressure increases.

Figure 5.9 presents the results for both analytical and numerical solution with a pretension of 10kPa and pressure-volume coupling subjected to external pressure values of 150kPa and 300kPa. Comparing the results of figures 5.9(a) and 5.8(a),

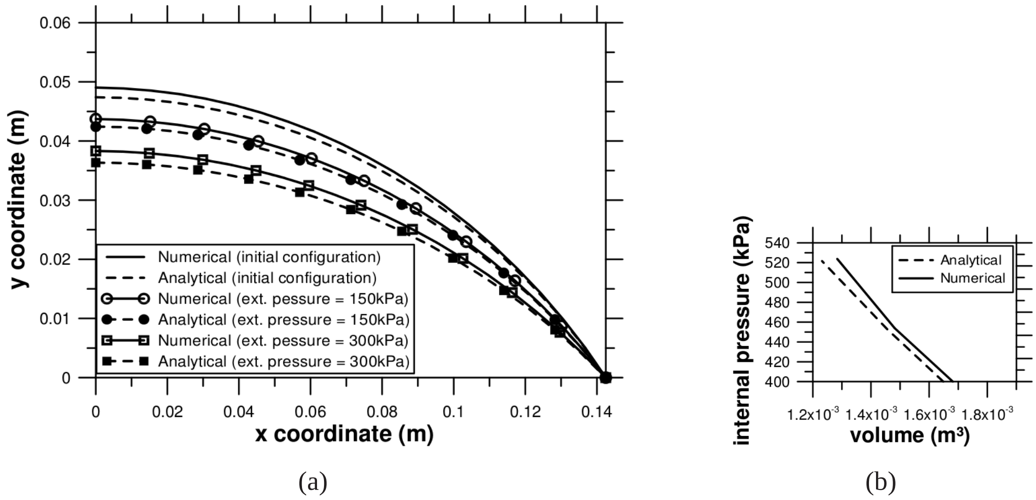


Figure 5.9: Analytical and numerical solution with a pretension of 10kPa and $\kappa = 1$ for the applied external pressure values of 150kPa and 300kPa: (a) deformed configuration and (b) pressure volume curve.

it is observed that the deformed configuration and consequently the volume of the circular membrane decreases for the case with a pretension of 10kPa. This difference is around 10%.

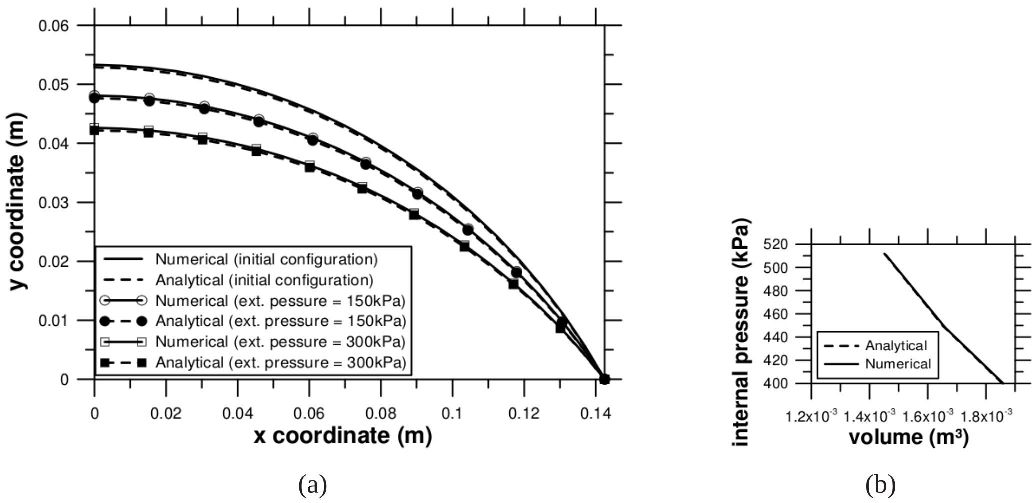


Figure 5.10: Analytical and numerical large strains solution without pretension and $\kappa = 1$ for applied external pressure values of 150kPa and 300kPa: (a) deformed configuration and (b) pressure volume curve.

The finite strain solution is shown in figure 5.10(a). The results are both for analytical and numerical solutions.

The results obtained with the analytical solution are similar to the numerical results, hightlithting that the difference between Fichter's solution and the numerical solution is due to the presence of large strains. This analytical solution also shows the importance of considering large strains by inflated membranes.