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A
Equations for non-particulate flow

A.1
Continuity

The continuity equation used in this work follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{A.1}$$

The associated residue follows:

$$R_C^i = \int_\Omega \psi_i \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) d\Omega \tag{A.2}$$

The non-zero Jacobian entries follow:

$$\frac{\partial R_C^i}{\partial u_j} = \int_\Omega \psi_i \frac{\partial \phi_j}{\partial x} d\Omega \tag{A.3}$$

$$\frac{\partial R_C^i}{\partial v_j} = \int_\Omega \psi_i \frac{\partial \phi_j}{\partial y} d\Omega \tag{A.4}$$

$$\frac{\partial R_C^i}{\partial w_j} = \int_\Omega \psi_i \frac{\partial \phi_j}{\partial z} d\Omega \tag{A.5}$$

The derivative of the Continuity's Residue with respect to pressure p_j is zero and it causes zero entries in the diagonal of the jacobian.

$$\frac{\partial R_C^i}{\partial p_j} = 0 \tag{A.6}$$

A.2
Momentum Conservation

Here follow the 3 components of the Momentum Conservation for non-particulate flow.

In x direction:

$$\rho_f\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=$$
$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]- \tag{A.7}$$
$$\frac{\partial p}{\partial x}+\rho_f g_x$$

In y direction:

$$\rho_f\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=$$
$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]- \tag{A.8}$$
$$\frac{\partial p}{\partial y}+\rho_f g_y$$

In z direction:

$$\rho_f\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=$$
$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}\right]- \tag{A.9}$$
$$\frac{\partial p}{\partial z}+\rho_f g_z$$

Their corresponding residues are $R^i_{m_x}$, $R^i_{m_y}$ and $R^i_{m_z}$:

$$\begin{aligned}R^i_{mx}=\int_\Omega\rho_f\phi_i\left(\frac{u-u_{old}}{\Delta t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{\partial\phi_i}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\\ \frac{\partial\phi_i}{\partial y}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial\phi_i}{\partial z}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)-\phi_i\,\rho_f\,g_x\,d\Omega\end{aligned} \tag{A.10}$$

$$\begin{aligned}R^i_{my}=\int_\Omega\rho_f\phi_i\left(\frac{v-v_{old}}{\Delta t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{\partial\phi_i}{\partial x}\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\\ \frac{\partial\phi_i}{\partial y}\left(-p+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial\phi_i}{\partial z}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)-\phi_i\,\rho_f\,g_y\,d\Omega\end{aligned} \tag{A.11}$$

$$\begin{aligned}R^i_{mz}=\int_\Omega\rho_f\phi_i\left(\frac{w-w_{old}}{\Delta t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\frac{\partial\phi_i}{\partial x}\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)+\\ \frac{\partial\phi_i}{\partial y}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{\partial\phi_i}{\partial z}\left(-p+2\mu\frac{\partial w}{\partial z}\right)-\phi_i\,\rho_f\,g_z\,d\Omega\end{aligned} \tag{A.12}$$

The non-zero Jacobian entries follow:

$$\frac{\partial R^i_{mx}}{\partial u_j} = \int_\Omega \rho_f \frac{\phi_i \phi_j}{\Delta t} + \rho_f \phi_i \left(\phi_j \frac{\partial u}{\partial x} + u \frac{\partial \phi_j}{\partial x} + v \frac{\partial \phi_j}{\partial y} + w \frac{\partial \phi_j}{\partial z} \right) + \frac{\partial \phi_i}{\partial x} 2\mu \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \mu \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial z} \, d\Omega$$

$$\frac{\partial R^i_{mx}}{\partial v_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial u}{\partial y} + \frac{\partial \phi_i}{\partial y} \mu \frac{\partial \phi_j}{\partial x} \, d\Omega \tag{A.13}$$

$$\frac{\partial R^i_{M_x}}{\partial w_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial u}{\partial z} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial x} \, d\Omega$$

$$\frac{\partial R^i_{mx}}{\partial P_j} = \int_\Omega - \frac{\partial \phi_i}{\partial x} \psi_j \, d\Omega$$

$$\frac{\partial R^i_{my}}{\partial u_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial v}{\partial x} + \frac{\partial \phi_i}{\partial x} \mu \frac{\partial \phi_j}{\partial y} \, d\Omega$$

$$\frac{\partial R^i_{my}}{\partial v_j} = \int_\Omega \rho_f \frac{\phi_i \phi_j}{\Delta t} + \rho_f \phi_i \left(\phi_j \frac{\partial v}{\partial y} + u \frac{\partial \phi_j}{\partial x} + v \frac{\partial \phi_j}{\partial y} + w \frac{\partial \phi_j}{\partial z} \right) + \frac{\partial \phi_i}{\partial x} \mu \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} 2\mu \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial z} \, d\Omega \tag{A.14}$$

$$\frac{\partial R^i_{my}}{\partial w_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial v}{\partial z} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial y} \, d\Omega$$

$$\frac{\partial R^i_{my}}{\partial P_j} = \int_\Omega - \frac{\partial \phi_i}{\partial y} \psi_j \, d\Omega$$

$$\frac{\partial R^i_{mz}}{\partial u_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial w}{\partial x} + \frac{\partial \phi_i}{\partial x} \mu \frac{\partial \phi_j}{\partial z} \, d\Omega$$

$$\frac{\partial R^i_{mz}}{\partial v_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial w}{\partial y} + \frac{\partial \phi_i}{\partial y} \mu \frac{\partial \phi_j}{\partial z} \, d\Omega$$

$$\begin{aligned} \frac{\partial R^i_{mz}}{\partial w_j} = \int_\Omega & \rho_f \frac{\phi_i \phi_j}{\Delta t} + \rho_f \phi_i \left(\phi_j \frac{\partial w}{\partial z} + u \frac{\partial \phi_j}{\partial x} + v \frac{\partial \phi_j}{\partial y} + w \frac{\partial \phi_j}{\partial z} \right) + \\ & \frac{\partial \phi_i}{\partial x} \mu \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \mu \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} 2\mu \frac{\partial \phi_j}{\partial z} \, d\Omega \end{aligned} \tag{A.15}$$

$$\frac{\partial R^i_{mz}}{\partial P_j} = \int_\Omega - \frac{\partial \phi_i}{\partial z} \psi_j \, d\Omega$$

B
Equations for particulate flow

B.1
Continuity

The continuity equation, its residue and the components of the Jacobian are exactly the same as the ones presented for non-particulate flow. See A.1.

B.2
Momentum Conservation

Here follow the 3 components of the Augmented Momentum Conservation for particulate flow.

In x direction:

$$\rho_f \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \\ \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] - \\ \frac{\partial p}{\partial x} + \rho_f g_x + \alpha\lambda_x - \mu\left(\frac{\partial^2 \lambda_x}{\partial x^2} + \frac{\partial^2 \lambda_x}{\partial y^2} + \frac{\partial^2 \lambda_x}{\partial z^2}\right) \tag{B.1}$$

In y direction:

$$\rho_f \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = \\ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] - \\ \frac{\partial p}{\partial y} + \rho_f g_y + \alpha\lambda_y - \mu\left(\frac{\partial^2 \lambda_y}{\partial x^2} + \frac{\partial^2 \lambda_y}{\partial y^2} + \frac{\partial^2 \lambda_y}{\partial z^2}\right) \tag{B.2}$$

In z direction:

$$\rho_f \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = \\ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}\right] - \\ \frac{\partial p}{\partial z} + \rho_f g_z + \alpha\lambda_z - \mu\left(\frac{\partial^2 \lambda_z}{\partial x^2} + \frac{\partial^2 \lambda_z}{\partial y^2} + \frac{\partial^2 \lambda_z}{\partial z^2}\right) \tag{B.3}$$

Their corresponding residues are $R^i_{M_x}$, $R^i_{M_y}$ and $R^i_{M_z}$, with uppercase "m" to differ from the non-particulate momentum:

$$\begin{aligned}R^i_{M_x} = \int_\Omega & \rho_f \phi_i \left(\frac{u - u_{old}}{\Delta t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) + \\ & \frac{\partial \phi_i}{\partial x} \left(-p + 2\mu \frac{\partial u}{\partial x} \right) + \frac{\partial \phi_i}{\partial y} \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial \phi_i}{\partial z} \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\ & - \phi_i \rho_f g_x - \phi_i \alpha \lambda_x - \mu \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \lambda_x}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \lambda_x}{\partial y} + \frac{\partial \phi_i}{\partial z}\frac{\partial \lambda_x}{\partial z} \right) d\Omega\end{aligned} \tag{B.4}$$

$$\begin{aligned}R^i_{M_y} = \int_\Omega & \rho_f \phi_i \left(\frac{v - v_{old}}{\Delta t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) + \\ & \frac{\partial \phi_i}{\partial x} \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial \phi_i}{\partial y} \left(-p + 2\mu \frac{\partial v}{\partial y} \right) + \frac{\partial \phi_i}{\partial z} \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ & - \phi_i \rho_f g_y - \phi_i \alpha \lambda_y - \mu \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \lambda_y}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \lambda_y}{\partial y} + \frac{\partial \phi_i}{\partial z}\frac{\partial \lambda_y}{\partial z} \right) d\Omega\end{aligned} \tag{B.5}$$

$$\begin{aligned}R^i_{M_z} = \int_\Omega & \rho_f \phi_i \left(\frac{w - w_{old}}{\Delta t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) + \\ & \frac{\partial \phi_i}{\partial x} \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) + \frac{\partial \phi_i}{\partial y} \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \frac{\partial \phi_i}{\partial z} \left(-p + 2\mu \frac{\partial w}{\partial z} \right) \\ & - \phi_i \rho_f g_z - \phi_i \alpha \lambda_z - \mu \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \lambda_z}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \lambda_z}{\partial y} + \frac{\partial \phi_i}{\partial z}\frac{\partial \lambda_z}{\partial z} \right) d\Omega\end{aligned} \tag{B.6}$$

The nonzero Jacobian elements for the Augmented Residues follow:

$$\frac{\partial R^i_{M_x}}{\partial u_j} = \int_\Omega \rho_f \frac{\phi_i \phi_j}{\Delta t} + \rho_f \phi_i \left(\phi_j \frac{\partial u}{\partial x} + u \frac{\partial \phi_j}{\partial x} + v \frac{\partial \phi_j}{\partial y} + w \frac{\partial \phi_j}{\partial z} \right) + \frac{\partial \phi_i}{\partial x} 2\mu \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \mu \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial z} \, d\Omega$$

$$\frac{\partial R^i_{M_x}}{\partial v_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial u}{\partial y} + \frac{\partial \phi_i}{\partial y} \mu \frac{\partial \phi_j}{\partial x} \, d\Omega$$

$$\frac{\partial R^i_{M_x}}{\partial w_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial u}{\partial z} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial x} \, d\Omega \tag{B.7}$$

$$\frac{\partial R^i_{M_x}}{\partial P_j} = \int_\Omega - \frac{\partial \phi_i}{\partial x} \psi_j \, d\Omega$$

$$\frac{\partial R^i_{M_x}}{\partial \lambda_{x_j}} = \int_\Omega - \phi_i \alpha \phi_j - \mu \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) d\Omega$$

$$\frac{\partial R^i_{M_y}}{\partial u_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial v}{\partial x} + \frac{\partial \phi_i}{\partial x} \mu \frac{\partial \phi_j}{\partial y} \, d\Omega$$

$$\frac{\partial R^i_{M_y}}{\partial v_j} = \int_\Omega \rho_f \frac{\phi_i \phi_j}{\Delta t} + \rho_f \phi_i \left(\phi_j \frac{\partial v}{\partial y} + u \frac{\partial \phi_j}{\partial x} + v \frac{\partial \phi_j}{\partial y} + w \frac{\partial \phi_j}{\partial z} \right) + \frac{\partial \phi_i}{\partial x} \mu \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} 2\mu \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial z} \, d\Omega$$

$$\frac{\partial R^i_{M_y}}{\partial w_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial v}{\partial z} + \frac{\partial \phi_i}{\partial z} \mu \frac{\partial \phi_j}{\partial y} \, d\Omega \tag{B.8}$$

$$\frac{\partial R^i_{M_y}}{\partial P_j} = \int_\Omega - \frac{\partial \phi_i}{\partial y} \psi_j \, d\Omega$$

$$\frac{\partial R^i_{M_y}}{\partial \lambda_{y_j}} = \int_\Omega - \phi_i \alpha \phi_j - \mu \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \frac{\partial \phi_j}{\partial z} \right) d\Omega$$

$$\frac{\partial R^i_{M_z}}{\partial u_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial w}{\partial x} + \frac{\partial \phi_i}{\partial x}\mu\frac{\partial \phi_j}{\partial z}\, d\Omega$$

$$\frac{\partial R^i_{M_z}}{\partial v_j} = \int_\Omega \rho_f \phi_i \phi_j \frac{\partial w}{\partial y} + \frac{\partial \phi_i}{\partial y}\mu\frac{\partial \phi_j}{\partial z}\, d\Omega$$

$$\begin{aligned}\frac{\partial R^i_{M_z}}{\partial w_j} = \int_\Omega \rho_f \frac{\phi_i\phi_j}{\Delta t} + \rho_f \phi_i \left(\phi_j \frac{\partial w}{\partial z} + u\frac{\partial \phi_j}{\partial x} + v\frac{\partial \phi_j}{\partial y} + w\frac{\partial \phi_j}{\partial z}\right) + \\ \frac{\partial \phi_i}{\partial x}\mu\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\mu\frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z}2\mu\frac{\partial \phi_j}{\partial z}\, d\Omega\end{aligned} \tag{B.9}$$

$$\frac{\partial R^i_{M_z}}{\partial P_j} = \int_\Omega -\frac{\partial \phi_i}{\partial z}\psi_j\, d\Omega$$

$$\frac{\partial R^i_{M_z}}{\partial \lambda_{z_j}} = \int_\Omega -\phi_i\alpha\phi_j - \mu\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z}\frac{\partial \phi_j}{\partial z}\right) d\Omega$$

B.3
Lagrange Multipliers Equations

The Lagrange Multipliers equations used in particulate flow are presented bellow:

In x direction:

$$\lambda_x = 0 \text{ in } \Omega_f \tag{B.10}$$

$$u = u_{P_k} + \left[\omega_{y_{P_k}}(z - z_{P_k}) - \omega_{z_{P_k}}(y - y_{P_k})\right] \text{ in } \Omega_{P_k} \tag{B.11}$$

In y direction:

$$\lambda_y = 0 \text{ in } \Omega_f \tag{B.12}$$

$$v = v_{P_k} + \left[\omega_{z_{P_k}}(x - x_{P_k}) - \omega_{x_{P_k}}(z - z_{P_k})\right] \text{ in } \Omega_{P_k} \tag{B.13}$$

In z direction:

$$\lambda_z = 0 \text{ in } \Omega_f \tag{B.14}$$

$$w = w_{P_k} + \left[\omega_{x_{P_k}}(y - y_{P_k}) - \omega_{y_{P_k}}(x - x_{P_k})\right] \text{ in } \Omega_{P_k} \tag{B.15}$$

The corresponding residues are $R^i_{\lambda_x}$, $R^i_{\lambda_y}$ and $R^i_{\lambda_z}$:

$$R^i_{\lambda_x} = \begin{cases} \int_{\Omega_f} \lambda_x \phi_i \, d\Omega_f \\ \int_{\Omega_{P_k}} (u - u_{P_k})\phi_i - \left[\omega_{y_{P_k}}(z - z_{P_k}) - \omega_{z_{P_k}}(y - y_{P_k})\right]\phi_i \, d\Omega_{P_k} \end{cases} \tag{B.16}$$

$$R^i_{\lambda_y} = \begin{cases} \int_{\Omega_f} \lambda_y \phi_i \, d\Omega_f \\ \int_{\Omega_{P_k}} (v - v_{P_k})\phi_i - \left[\omega_{z_{P_k}}(x - x_{P_k}) - \omega_{x_{P_k}}(z - z_{P_k})\right] \phi_i \, d\Omega_{P_k} \end{cases} \tag{B.17}$$

$$R^i_{\lambda_z} = \begin{cases} \int_{\Omega_f} \lambda_z \phi_i \, d\Omega_f \\ \int_{\Omega_{P_k}} (w - w_{P_k})\phi_i - \left[\omega_{x_{P_k}}(y - y_{P_k}) - \omega_{y_{P_k}}(x - x_{P_k})\right] \phi_i \, d\Omega_{P_k} \end{cases} \tag{B.18}$$

The non-zero Jacobian entries for the λ residues follow:

$$\frac{\partial R^i_{\lambda_x}}{\partial \lambda^j_x} = \frac{\partial R^i_{\lambda_y}}{\partial \lambda^j_y} = \frac{\partial R^i_{\lambda_z}}{\partial \lambda^j_z} = \begin{cases} \int_{\Omega_f} \phi_i \phi_j \, d\Omega_f \\ \int_{\Omega_{P_k}} 0 \, d\Omega_{P_k} \end{cases} \tag{B.19}$$

$$\frac{\partial R^i_{\lambda_x}}{\partial u_j} = \frac{\partial R^i_{\lambda_y}}{\partial v_j} = \frac{\partial R^i_{\lambda_z}}{\partial w_j} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} \phi_i \phi_j \, d\Omega_{P_k} \end{cases} \tag{B.20}$$

$$\frac{\partial R^i_{\lambda_x}}{\partial u_{P_k}} = \frac{\partial R^i_{\lambda_y}}{\partial v_{P_k}} = \frac{\partial R^i_{\lambda_z}}{\partial w_{P_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} -\phi_i \, d\Omega_{P_k} \end{cases} \tag{B.21}$$

$$\frac{\partial R^i_{\lambda_x}}{\partial \omega_{yP_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} -\phi_i (z - z_{P_k}) \, d\Omega_{P_k} \end{cases} \tag{B.22}$$

$$\frac{\partial R^i_{\lambda_x}}{\partial \omega_{zP_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} \phi_i (y - y_{P_k}) \, d\Omega_{P_k} \end{cases} \tag{B.23}$$

$$\frac{\partial R^i_{\lambda_y}}{\partial \omega_{zP_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} -\phi_i (x - x_{P_k}) \, d\Omega_{P_k} \end{cases} \tag{B.24}$$

$$\frac{\partial R^i_{\lambda_y}}{\partial \omega_{xP_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} \phi_i (z - z_{P_k}) \, d\Omega_{P_k} \end{cases} \tag{B.25}$$

$$\frac{\partial R^i_{\lambda_z}}{\partial \omega_{xP_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} -\phi_i (y - y_{P_k}) \, d\Omega_{P_k} \end{cases} \tag{B.26}$$

$$\frac{\partial R^i_{\lambda_z}}{\partial \omega_{yP_k}} = \begin{cases} \int_{\Omega_f} 0 \, d\Omega_f \\ \int_{\Omega_{P_k}} \phi_i (x - x_{P_k}) \, d\Omega_{P_k} \end{cases} \tag{B.27}$$

B.4
Equations of the Rigid Body Dynamics

The components of the residue of the linear velocity of the particle P_k are given by $R_u^{P_k}$, $R_v^{P_k}$ and $R_w^{P_k}$:

$$R_u^{P_k} = \int_{\Omega_{P_k}} (\rho_{P_k} - \rho_f)\frac{\partial u_{P_k}}{\partial t} - \rho_{P_k} g_x + \frac{\partial p}{\partial x} + \alpha\lambda_x \, d\Omega_{P_k} \tag{B.28}$$

$$R_v^{P_k} = \int_{\Omega_{P_k}} (\rho_{P_k} - \rho_f)\frac{\partial v_{P_k}}{\partial t} - \rho_{P_k} g_y + \frac{\partial p}{\partial y} + \alpha\lambda_y \, d\Omega_{P_k} \tag{B.29}$$

$$R_w^{P_k} = \int_{\Omega_{P_k}} (\rho_{P_k} - \rho_f)\frac{\partial w_{P_k}}{\partial t} - \rho_{P_k} g_z + \frac{\partial p}{\partial z} + \alpha\lambda_z \, d\Omega_{P_k} \tag{B.30}$$

The components of the residue of the angular velocity of the particle P_k are given by $R_{\omega_x}^{P_k}$, $R_{\omega_y}^{P_k}$ and $R_{\omega_z}^{P_k}$:

$$R_{\omega_x}^{P_k} = \int_{\Omega_{P_k}} \omega_{x_{P_k}} - \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) d\Omega_{P_k} \tag{B.31}$$

$$R_{\omega_y}^{P_k} = \int_{\Omega_{P_k}} \omega_{y_{P_k}} - \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) d\Omega_{P_k} \tag{B.32}$$

$$R_{\omega_z}^{P_k} = \int_{\Omega_{P_k}} \omega_{z_{P_k}} - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) d\Omega_{P_k} \tag{B.33}$$

The non-zero Jacobian entries for the linear velocities follow:

$$\frac{\partial R_u^{P_k}}{\partial u_{P_k}} = \frac{\partial R_v^{P_k}}{\partial v_{P_k}} = \frac{\partial R_w^{P_k}}{\partial w_{P_k}} = \int_{\Omega_{P_k}} (\rho_{P_k} - \rho_f)\,\frac{1}{\Delta t}\, d\Omega_{P_k} \tag{B.34}$$

$$\frac{\partial R_u^{P_k}}{\partial \lambda_{x_j}} = \frac{\partial R_v^{P_k}}{\partial \lambda_{y_j}} = \frac{\partial R_w^{P_k}}{\partial \lambda_{z_j}} = \int_{\Omega_{P_k}} \alpha\phi_j \, d\Omega_{P_k} \tag{B.35}$$

$$\frac{\partial R_u^{P_k}}{\partial p_j} = \int_{\Omega_{P_k}} \frac{\partial \psi_j}{\partial x}\, d\Omega_{P_k} \tag{B.36}$$

$$\frac{\partial R_v^{P_k}}{\partial p_j} = \int_{\Omega_{P_k}} \frac{\partial \psi_j}{\partial y}\, d\Omega_{P_k} \tag{B.37}$$

$$\frac{\partial R_w^{P_k}}{\partial p_j} = \int_{\Omega_{P_k}} \frac{\partial \psi_j}{\partial z}\, d\Omega_{P_k} \tag{B.38}$$

The non-zero Jacobian entries for the angular velocities follow:

$$\frac{\partial R_{\omega_x}^{P_k}}{\partial \omega_{x_{P_k}}} = \frac{\partial R_{\omega_y}^{P_k}}{\partial \omega_{y_{P_k}}} = \frac{\partial R_{\omega_z}^{P_k}}{\partial \omega_{z_{P_k}}} = \int_{\Omega_{P_k}} 1 \, d\Omega_{P_k} \tag{B.39}$$

$$\frac{\partial R_{\omega_x}^{P_k}}{\partial v_j} = \int_{\Omega_{P_k}} \frac{1}{2}\frac{\partial \phi_j}{\partial z}\, d\Omega_{P_k} \tag{B.40}$$

$$\frac{\partial R_{\omega_x}^{P_k}}{\partial w_j} = \int_{\Omega_{P_k}} -\frac{1}{2}\frac{\partial \phi_j}{\partial y}\, d\Omega_{P_k} \tag{B.41}$$

$$\frac{\partial R_{\omega_y}^{P_k}}{\partial u_j} = \int_{\Omega_{P_k}} -\frac{1}{2}\frac{\partial \phi_j}{\partial z}\, d\Omega_{P_k} \tag{B.42}$$

$$\frac{\partial R_{\omega_y}^{P_k}}{\partial w_j} = \int_{\Omega_{P_k}} \frac{1}{2}\frac{\partial \phi_j}{\partial x}\, d\Omega_{P_k} \tag{B.43}$$

$$\frac{\partial R_{\omega_z}^{P_k}}{\partial u_j} = \int_{\Omega_{P_k}} \frac{1}{2}\frac{\partial \phi_j}{\partial y}\, d\Omega_{P_k} \tag{B.44}$$

$$\frac{\partial R_{\omega_z}^{P_k}}{\partial v_j} = \int_{\Omega_{P_k}} -\frac{1}{2}\frac{\partial \phi_j}{\partial x}\, d\Omega_{P_k} \tag{B.45}$$