2
Preliminaries

In this section we collect several standard definitions and facts from Linear Algebra, Differential Topology and Dynamical Systems that will be used later.

2.1
Basic facts about vector fields and flows

One important tool in the analysis of the local structure of periodic orbits is the Poincaré first return map, a discrete dynamical system defined in a cross section that inherits local properties of the flow close to a periodic orbit. In this work we use a more general definition of the Poincaré map, which allows the map to be a first hit map between two cross sections. We also allow the cross sections to be general codimension 1 submanifolds with boundary. It is convenient to impose a certain *compatibility* condition on those submanifolds:

Definition 2.1.1 (compatibility) *Given $X \in \mathfrak{X}^1(M)$ and its induced flow $\{\varphi^t\}_t$, we say that two codimension 1 submanifolds with boundary Σ_1 and Σ_2 are* compatible *if their union is still a submanifold with boundary, if they are both transverse to X and the following holds:*

$$\inf\{t > 0 : \varphi^t(x) \in \Sigma_2\} \le \inf\{t > 0 : \varphi^t(x) \in \Sigma_1\},\ \forall x \in \Sigma_1 \tag{2.1}$$

$$\inf\{t > 0 : \varphi^{-t}(y) \in \Sigma_1\} \le \inf\{t > 0 : \varphi^{-t}(y) \in \Sigma_2\},\ \forall y \in \Sigma_2. \tag{2.2}$$

So compatibility forbids the situation of Figure 2.1. Also, note that the above definition does not exclude the possibility of Σ_1 being equal to Σ_2, since we want to consider the Poincaré first return map as a particular case of the map we are about to construct.

Definition 2.1.2 (hitting-time) *Let Σ_1 and Σ_2 be compatible cross sections. Then we define the* hitting-time *function $\tau : \Sigma_1 \to \mathbb{R}^+ \cup \{\infty\}$ by*

$$\tau(x) = \inf\{t > 0 : \varphi^t(x) \in \Sigma_2\}$$

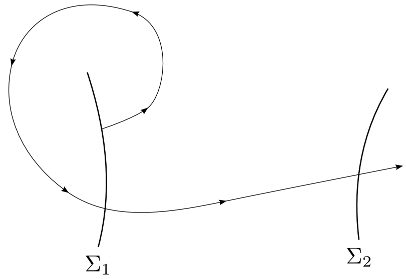

Figure 2.1: Non-compatible cross-sections.

and its backwards version $\tau' : \Sigma_2 \to \mathbb{R}^+ \cup \{\infty\}$ *by*

$$\tau'(y) = \inf\{t > 0 : \varphi^{-t}(y) \in \Sigma_2\}.$$

Here $\tau(x) = \infty$ *(resp.* $\tau'(y) = \infty$*) means that the future orbit of* x *(resp. the past orbit of* y*) does not intersect* Σ_2 *(resp.* Σ_1*).*

Proposition 2.1.3 (Poincaré Map) *Let* Σ_1 *and* Σ_2 *be two compatible cross sections and let* σ_1 *and* σ_2 *be their respective induced Riemannian measures. Consider the following subsets of the cross sections:*

$$\tilde{\Sigma}_1 = \{x \in \Sigma_1 \backslash \partial\Sigma_1 : \tau(x) < \infty,\ \varphi^{\tau(x)}(x) \in \Sigma_2 \backslash \partial\Sigma_2\}$$
$$\tilde{\Sigma}_2 = \{y \in \Sigma_2 \backslash \partial\Sigma_2 : \tau'(y) < \infty,\ \varphi^{-\tau'(y)}(y) \in \Sigma_1 \backslash \partial\Sigma_1\}.$$

Then:

1. $\tilde{\Sigma}_1$ *is open in* Σ_1*;*

2. $\tilde{\Sigma}_2$ *is open in* Σ_2*;*

3. $\tau|_{\tilde{\Sigma}_1}$ *and* $\tau'|_{\tilde{\Sigma}_2}$ *are* C^1 *maps;*

4. *the map*
$$\begin{array}{rccl} f\colon & \tilde{\Sigma}_1 & \to & \tilde{\Sigma}_2 \\ & x & \mapsto & \varphi^{\tau(x)}(x) \end{array}$$
is a diffeomorphism, with inverse
$$\begin{array}{rccl} f^{-1}\colon & \tilde{\Sigma}_2 & \to & \tilde{\Sigma}_1 \\ & y & \mapsto & \varphi^{-\tau'(y)}(y) \end{array};$$

5. *for* σ_1*-a.e.* $x \in \Sigma_1$*, either* $\tau(x) = \infty$ *or* $x \in \tilde{\Sigma}_1$*.*

Proof: Parts 1, 2 and 3 are easy consequences of the (long) flow-box theorem (Proposition 1.1 in Chapter 3 of [PdM]). Notice that the compatibility of the cross sections guarantees that f is one-to-one. Its inverse is given by

$$f^{-1}(y) = \varphi^{-\tau'(y)}(y).$$

It follows from part 3 that f and f^{-1} are C^1 maps, thus proving part 4.

For the proof of part 5, define the following subsets:

$$F_i = \bigcup_{t\in\mathbb{R}} \varphi^t(\partial\Sigma_i), \quad (i = 1, 2).$$

Then $F_i \subset M$ is an immersed codimension 1 submanifold transverse to Σ_1. Therefore the intersection $F_i \cap \Sigma_1$ is an immersed codimension 2 submanifold of Σ_1, and in particular it has zero σ_1 measure. Noticing that $x \in \Sigma_1 \setminus (F_1 \cup F_2)$ implies that either $\tau(x) = \infty$ or $x \in \tilde{\Sigma}_1$, the proof of the proposition is concluded. ∎

The diffeomorphism $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ defined in the previous proposition will be called *Poincaré map.*

Corollary 2.1.4 *Let Σ_1 and Σ_2 be two compatible cross sections and let $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ be the induced Poincaré map. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset \tilde{\Sigma}_1$ is a measurable set with $\sigma_1(A) < \delta$, then*

$$m\left(\bigcup_{p\in A}\ \bigcup_{t\in[0,\tau(p)]} \varphi^t(p)\right) < \epsilon.$$

Proof: It suffices to note that

$$\sigma_*(A) = m\left(\bigcup_{p\in A}\ \bigcup_{t\in[0,\tau(p)]} \varphi^t(p)\right)$$

defines a measure on $\tilde{\Sigma}_1$ which is absolutely continuous with respect to σ_1. ∎

Notice the following consequence of long flow-box theorem:

Remark 2.1.5 *Let $t_0 > 0$ and let $p \in M$ be a non-periodic point or a periodic point with period bigger then t_0. Suppose Σ_1 and Σ_2 are cross sections (i.e., codimension 1 submanifolds tranverse to the flow) such that $p \in \Sigma_1 \setminus \partial\Sigma_1$ and $\varphi^{t_0}(p) \in \Sigma_2 \setminus \partial\Sigma_2$. Then there exist closed neighborhoods Σ_1^* and Σ_2^* of p and $\varphi^{t_0}(p)$ in Σ_1 and Σ_2 respectively that are compatible cross-sections. Moreover, $\tau(x) < \infty$ for all $x \in \Sigma_1^*$.*

We say that the Poincaré map as in Remark 2.1.5 is *based* on the orbit of p with respect to the base time t_0 and denote its hitting time by τ_{X,p,t_0}. Depending on the context, the vector field X, the point p and/or the base time t_0 that define the hitting-time map with respect to a segment of orbit will be omitted from the notation, yielding τ_X, τ_{t_0} or simply τ. We denote this Poincaré map by

$$\begin{array}{rccl} \Phi_{t_0}: & \Sigma_1^* & \to & \Sigma_2^* \\ & x & \mapsto & \Phi_{t_0}(x) = \varphi^{\tau(x)}(x). \end{array} \tag{2.3}$$

Remark 2.1.6 *Since the hitting-time map is C^1 and, therefore, continuous, we have that for all $\epsilon > 0$ there exists a neighborhood $V \subset \Sigma_1^*$ of p such that*

$$|t_0 - \tau_{t_0}(x)| < \epsilon$$

for all $x \in V \cap \Sigma_1^*$.

Recall that if $G : U_1 \to U_2$ is a diffeomorphism and $X \in \mathfrak{X}^1(U_1)$ is a vector field, we define its *push-forward* $F_*X \in \mathfrak{X}^1(U_2)$ by

$$(F_*X)(z) \equiv DG(G^{-1}(z)) \cdot X(G^{-1}(z)).$$

The flows of the two vector fields are conjugate by the diffeomorphism G.

In Section 3, we will consider the Poincaré map with respect to some well-chosen sections with properties. In that construction, we will make use of "adapted" coordinates around a hyperbolic singularity (i.e., a fixed point of the flow), which are given by next lemma. The *stable* (resp. *unstable*) *index* of a hyperbolic singularity is the dimension of its stable (resp. unstable) manifold; in particular the sum of the indices equals $d = \dim M$.

Lemma 2.1.7 (adapted coordinates) *Let $X \in \mathfrak{X}^1(M)$. Suppose $p \in M$ is a hyperbolic singularity of X, and let s and u be respectively the stable and unstable indices. Then there exist*

- *a chart $F : U \to V$, where U and V are open neighborhoods of $p \in M$ and $0 \in \mathbb{R}^d$, respectively;*
- *constants $\Lambda > \lambda > 0$;*

with the following properties:

1. *$F(p) = 0$.*
2. *The local stable (resp. unstable) manifold at p is mapped by F into $\mathbb{R}^s \times \{0\}$ (resp. $\{0\} \times \mathbb{R}^u$).*

3. *Suppose $x : \mathbb{R} \to M$ is an orbit of the flow generated by X and $I \subset \mathbb{R}$ is an interval such that $x(I) \subset U$. For $t \in I$, write*

$$F(x(t)) = \big(y_{\rm s}(t), y_{\rm u}(t)\big) \quad \text{with} \quad y_{\rm s}(t) \in \mathbb{R}^s,\ y_{\rm u}(t) \in \mathbb{R}^u\,.$$

Then for all t_0, $t_1 \in I$ with $t_0 < t_1$ we have:

$$e^{-\Lambda(t_1-t_0)}\|y_{\rm s}(t_0)\| \le \|y_{\rm s}(t_1)\| \le e^{-\lambda(t_1-t_0)}\|y_{\rm s}(t_0)\|\,, \tag{2.4}$$
$$e^{\lambda(t_1-t_0)}\|y_{\rm u}(t_0)\| \le \|y_{\rm u}(t_1)\| \le e^{\Lambda(t_1-t_0)}\|y_{\rm u}(t_0)\|\,, \tag{2.5}$$

where $\|\cdot\|$ denotes Euclidean norm.

Lemma 2.1.7 is probably well-known, but being without a precise reference, we will provide a proof. We begin with the following linear algebraic fact:

Lemma 2.1.8 *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map without purely imaginary eigenvalues. Let $E^{\rm s}$ (resp. $E^{\rm u}$) be the generalized eigenspace corresponding to eigenvalues of negative (resp. positive) real part. Then there exists an "adapted" inner product $\langle\cdot,\cdot\rangle_{\rm a}$ on $\mathbb{R}^d$ and constants $\Lambda > \lambda > 0$ such that, for all $v_{\rm s} \in E^{\rm s}$, $v_{\rm u} \in E^{\rm u}$ we have:*

$$\langle v_{\rm s}, v_{\rm u}\rangle_{\rm a} = 0\,, \tag{2.6}$$
$$-\Lambda\|v_{\rm s}\|_{\rm a}^2 \le \langle Lv_{\rm s}, v_{\rm s}\rangle_{\rm a} \le -\lambda\|v_{\rm s}\|_{\rm a}^2\,, \tag{2.7}$$
$$\lambda\|v_{\rm u}\|_{\rm a}^2 \le \langle Lv_{\rm u}, v_{\rm u}\rangle_{\rm a} \le \Lambda\|v_{\rm u}\|_{\rm a}^2\,. \tag{2.8}$$

where $\|v\|_{\rm a}^2 = \langle v, v\rangle_{\rm a}$.

Proof: First consider the case where all eigenvalues of L have negative real part. Then the exponential matrix e^L has spectral radius $\rho < 1$. Let $\|\cdot\|$ be the Euclidean norm. By the spectral radius theorem (Gelfand formula), we have $\lim_{t\to+\infty} \frac{1}{t}\log\|e^{tL}\| = \log\rho < 0$. Therefore the following expression defines a new norm:

$$\|v\|_{\rm a}^2 = \int_0^\infty \|e^{tL}\cdot v\|^2\,dt\,.$$

It is clear that this norm corresponds to an inner product $\langle\cdot,\cdot\rangle_{\rm a}$. Notice that

$$s \ge 0 \ \Rightarrow\ \|e^{sL}\cdot v\|_{\rm a}^2 = \int_s^\infty \|e^{tL}\cdot v\|^2\,dt\,.$$

In particular,

$$\left.\frac{d}{ds}\right|_{s=0} \|e^{sL}\cdot v\|_{\rm a}^2 = -\|v\|^2\,.$$

On the other hand, the same derivative can be computed as

$$\frac{d}{ds}\bigg|_{s=0} \langle e^{sL} \cdot v, e^{sL} \cdot v\rangle_{\mathrm{a}} = 2\langle Lv, v\rangle_{\mathrm{a}} \, .$$

Thus $\langle Lv, v\rangle_{\mathrm{a}} = -\frac{1}{2}\|v\|^2$, which is between $-\Lambda\|v\|_{\mathrm{a}}^2$ and $-\lambda\|v\|_{\mathrm{a}}^2$ for some constants $\Lambda > \lambda > 0$.

We proved the lemma in the particular case where all eigenvalues of L have negative real part. The general case of the lemma follows by considering the restrictions $L|E^{\mathrm{s}}$ and $(-L)|E^{\mathrm{u}}$ and taking the orthogonal sum inner product. ∎

Remark 2.1.9 *All inner products on $\mathbb{R}^d$ coincide modulo a linear change of coordinates. Therefore, in the situation of Lemma 2.1.8 we can find an invertible linear map $S : \mathbb{R}^d \to \mathbb{R}^d$ such that if L, E^{s} and E^{u} are replaced with SLS^{-1}, $S(E^{\mathrm{s}})$, $S(E^{\mathrm{u}})$, then the relations* (2.6), (2.7), (2.8) *hold with $\langle\cdot,\cdot\rangle_{\mathrm{a}}$ being the Euclidean inner product.*

Proof of Lemma 2.1.7: By changing coordinates, we can assume that the vector field X is defined on a neighborhood of $p = 0$ in $\mathbb{R}^d$. Let E^{s} and E^{u} denote the stable and unstable subspaces. As a trivial consequence of the stable manifold theorem (see for example [PdM, pp.88–89]), we can change coordinates again so that the local stable and unstable manifolds are contained in the vector subspaces E^{s} and E^{u}, respectively. By applying the linear change of coordinates given by Remark 2.1.9, we can assume that there are constants $\Lambda > \lambda > 0$ such that relations (2.6), (2.7), (2.8) hold for $L = DX(0)$, with $\langle\cdot,\cdot\rangle_{\mathrm{a}}$ being the Euclidean inner product. By a final change of coordinates using an orthogonal linear map, we can assume that $E^{\mathrm{s}} = \mathbb{R}^s \times \{0\}$ and $E^{\mathrm{u}} = \{0\} \times \mathbb{R}^u$.

In coordinates $(y_{\mathrm{s}}, y_{\mathrm{u}}) \in \mathbb{R}^s \times \mathbb{R}^u$, we write

$$X(y_{\mathrm{s}}, y_{\mathrm{u}}) = \big(X_{\mathrm{s}}(y_{\mathrm{s}}, y_{\mathrm{u}}), X_{\mathrm{u}}(y_{\mathrm{s}}, y_{\mathrm{u}})\big) \, .$$

Then we have

$$X_{\mathrm{s}}(y_{\mathrm{s}}, 0) = 0 \, , \quad X_{\mathrm{u}}(0, y_{\mathrm{u}}) = 0 \, .$$

Fix a positive $\epsilon < \lambda$. Then for every $(y_{\mathrm{s}}, y_{\mathrm{u}})$ sufficiently close to $(0, 0)$, we have

$$\begin{aligned} \big\|X_{\mathrm{s}}(y_{\mathrm{s}}, y_{\mathrm{u}}) - L(y_{\mathrm{s}}, 0)\big\| &\le \epsilon\|y_{\mathrm{s}}\| \, , \\ \big\|X_{\mathrm{u}}(y_{\mathrm{s}}, y_{\mathrm{u}}) - L(0, y_{\mathrm{u}})\big\| &\le \epsilon\|y_{\mathrm{u}}\| \, . \end{aligned}$$

We reduce the chart domain so that these properties are satisfied. Now assume that $t \in I \mapsto (y_{\mathrm{s}}(t), y_{\mathrm{u}}(t))$ is a trajectory of the flow contained in this chart domain. Then

$$\left.\frac{d}{dt}\right|_{t=0} \|y_{\mathrm{s}}(t)\|^2 = 2\big\langle X_{\mathrm{s}}(y_{\mathrm{s}}(t), y_{\mathrm{u}}(t)), (y_{\mathrm{s}}(t), 0)\big\rangle$$
$$\leq 2(-\lambda + \epsilon)\|y_{\mathrm{s}}(t)\|^2$$

This implies that the second inequality in (2.4) holds with $\lambda - \epsilon$ in the place of λ. The remaining inequalities are proven similarly (with Λ replaced by $\Lambda + \epsilon$). ■

Recall that the main part of the proof of the main result is to perturb a given vector field so that it has the δ-crushing property. Actually we will perform a few successive perturbations, each one preparing the ground for the next one. In this regard, the following fact will be useful:

Proposition 2.1.10 *The set $\mathfrak{I} \subset \mathfrak{X}^r(M)$ of vector fields such that all periodic orbits are hyperbolic (and isolated) is a C^1-open and dense set.*

Proof: This proposition is a intermediate step of the proof of the Kupka–Smale Theorem and can be found for example in [PdM, p.115]. ■

2.2 Non-Conformality

If L is a linear isomorphism between inner-product vector spaces, the *non-conformality* of L is

$$\mathrm{NC}(L) \equiv \|L\|\,\|L^{-1}\|.$$

This quantity measures how much L can distort angles, in fact:

$$\frac{1}{\|L\| \cdot \|L^{-1}\|} \leq \frac{\sin(\angle(Lu, Lv))}{\sin(\angle(u, v))} \leq \|L\| \cdot \|L^{-1}\|. \tag{2.9}$$

See [BV, Lemma 2.7] for a proof of (2.9).

Next Proposition is a simple Linear Algebra fact and follows from the definition of matrix induced norm (See Figure 2.2 for an illustrative idea of the proof).

Proposition 2.2.1 *Let L be an invertible map and let $\mathcal{B}(r)$ be the Euclidean ball with radius r centered in the origin. If $r_1 < r_2$ are such that*

$$B(r_1) \subset L(B(r)) \subset B(r_2), \tag{2.10}$$

then $r_2 > r_1 \cdot NC(L)$. Moreover, $r_2 > r \cdot \max\{\|L\|, \|L^{-1}\|\}$ satisfies (2.10), with $r_2 = r_1 \cdot \mathrm{NC}(L)$.

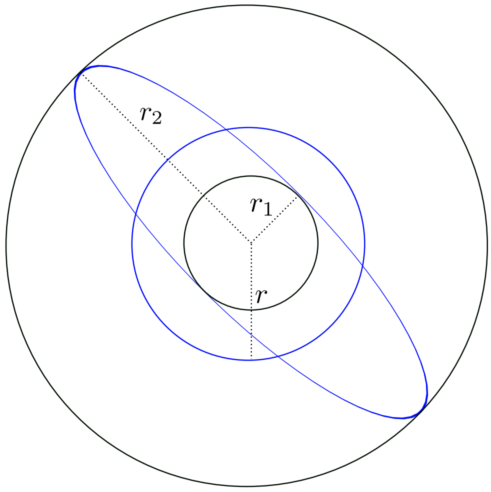

Figure 2.2: The image of an Euclidean ball by a linear invertible map L is incribed in a sphere with radius $r_2 = \|L\|r$ and circumscribed on a sphere with radius $r_1 = \|L^{-1}\|^{-1}r$.

2.3
Linear Cocycles

The word "cocycle" can be found in Mathematics with very different meanings and the term seems to have been borrowed from Algebraic Topology. Let us see its Dynamical Systems' definition.

Definition 2.3.1 *A* flow *on a manifold M is an action of $\mathbb{R}$ by diffeomorphisms, i.e., a collection of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$ such that $\varphi^{t+s} = \varphi^t \circ \varphi^s$. We also ask the joint map $(t,x) \in \mathbb{R} \times M \mapsto \varphi^t(x) \in M$ to be continuous.*

Definition 2.3.2 *Let $\varphi^t : M \to M$ be a flow on a smooth manifold M and let $\pi : E \to M$ be a fiber bundle over M. A* cocycle *over the flow φ^t is a flow*

$$F^t : E \to E$$

such that $\pi \circ F^t = \varphi^t \circ \pi$.

Notice that the restriction of F^t to the fiber $\pi^{-1}(x)$ is a diffeomorphism onto the fiber $\pi^{-1}(\varphi^t x)$, which we denote by $A^t(x) : \pi^{-1}(x) \to \pi^{-1}(\varphi^t x)$. The following properties hold:

1. $A^0(x) = Id$;
2. $A^{t+s}(x) = A^s(\varphi^t(x))A^t(x)$. (cocycle condition).

A special case is that of *linear cocycles*:

Definition 2.3.3 *Let $\varphi^t : M \to M$ be a flow on a smooth manifold M. Let $\pi : E \to M$ be a vector bundle over M. A cocycle $F^t : E \to E$ is called a* linear cocycle *if the maps $A^t(x)$ between fibers are linear.*

In the case the vector bundle E is trivial, i.e., $E = M \times \mathbb{R}^n$, then the linear cocycle takes the form:

$$F^t(x, v) = (\varphi^t(x), A^t(x)v),$$

where $A^t(x) \in GL(n, \mathbb{R})$ for all $x \in M$. Conversely, if A^t is a family of linear maps with $A^0 = Id$ and satisfying the cocycle condition then we can define a linear cocycle by the formula above. The family of linear maps $A^t : M \to GL(n, \mathbb{R})$ will be ambiguously called "cocycle".

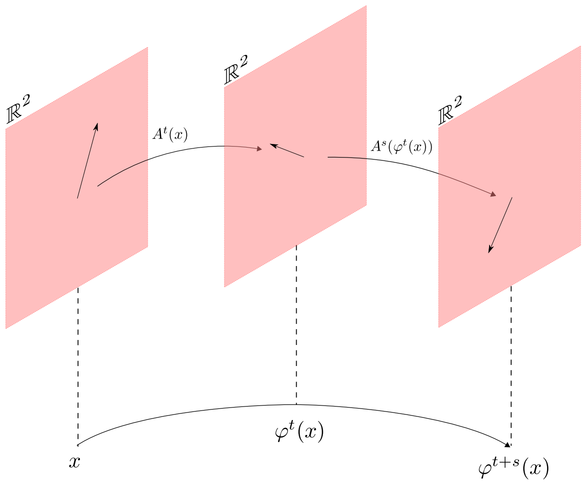

Figure 2.3: A linear cocycle over the flow $\{\varphi^t\}$.

Definition 2.3.4 *Let $A^t : M \to GL(n, \mathbb{R})$ be a cocycle which is differentiable in the t parameter. The (infinitesimal) generator of A^t is the function $a : M \to GL(n, \mathbb{R})$, given by*

$$a(x) = \frac{\partial}{\partial t} A^t(x) \bigg|_{t=0} .$$

Remark 2.3.5 *The name generator in the previous definition comes from the fact that a cocycle may be generated by a non-autonomous differential equation:*

$$\frac{\partial}{\partial t} A^t(x) = a(\varphi^t(x)) A^t(x),$$

with initial condition $A^0(x) = Id$.

Proposition 2.3.6 *Let $A^t : M \to GL(n,\mathbb{R})$ be a cocycle with generator $a : M \to GL(n,\mathbb{R})$. Then we have:*

1. $\|A^t(p)\| \le e^{C|t|}$;
2. $\|A^t(p) - Id\| \le e^{C|t|} - 1$,

where $C = \sup_{x\in M} \|a(x)\|$.

Proof: In order to prove part 1, define $f(t) = \|A^t(x)\|$ and note that

$$\begin{aligned} |f'(t)| &\le \left\| \frac{\partial}{\partial t} A^t(x) \right\| \\ &= \|a(\varphi^t(x))A^t(x)\| \\ &\le Cf(t). \end{aligned}$$

That is, $|(\log f(t))'| \le C$. Since $f(0) = \|Id\| = 1$, we have $f(t) \le e^{C|t|}$, as we wanted to show.

The proof of part 2 is analogous. Let us now consider $B^t = A^t - Id$. Then we have

$$\frac{\partial}{\partial t} B^t(x) = a(\varphi^t(x))(Id + B^t(x)).$$

Defining the function $g(t) = \|B^t(x)\|$, we have that

$$\begin{aligned} |g'(t)| &\le \left\| \frac{\partial}{\partial t} B^t(x) \right\| \\ &\le C(1 + g(t)). \end{aligned}$$

The solution of the ODE:

$$\begin{aligned} &h'(t) = C(1 + h(t)), \\ &h(0) = 0, \end{aligned}$$

is $h(t) = e^{Ct} - 1$. Thus if $t > 0$ then $g(t) \le \int_0^t |g'| \le h(t) = e^{Ct} - 1$, and analogously for $t < 0$. ■

Every linear cocycle in $GL(n,\mathbb{R})$ over a flow φ^t induces a cocycle in $\mathbb{R}$, by taking the determinant of the matrix $A^t(x)$. The precise statement of this well known result is given by the following proposition and its proof can be found for example in [CL, Theorem I.7.3].

Proposition 2.3.7 *Let $A : \mathbb{R} \times M \to GL(n, \mathbb{R})$ be a cocycle over the flow $\varphi : \mathbb{R} \times M \to M$ with generator $G : M \to GL(n, \mathbb{R})$. Then the function $f : \mathbb{R} \times M \to \mathbb{R}$, defined by*

$$f(t, p) = \det A^t(p)$$

is a linear cocycle in $\mathbb{R}$ over the same flow. Moreover its generator $g : M \to \mathbb{R}$ is given by

$$g(p) = \operatorname{tr} G(p). \tag{2.11}$$

Consider $X \in \mathfrak{X}^1(M)$. Let us see some natural examples of linear cocycles over the flow φ^t generated by a vector field X. The first one is the *derivative cocycle*:

$$T_x M \to T_{\varphi^t(x)} M$$

$$u \mapsto D\varphi^t(x)u,$$

The cocycle condition is a direct consequence of the chain rule.

The second example is the *linear Poincaré flow.* Let $R(X) \subset M$ be the set of regular points in M, that is,

$$R(X) = \{x \in M : X(x) \neq 0\}.$$

Let us define the normal bundle $N_{R(X)}$ associated to X. For each $x \in R(X)$, let N_x be the orthogonal complement of $X(x)$ in $T_x M$. This is a fiber of a vector bundle over $R(X)$, which is a subbundle of $T_{R(X)} M$.

Definition 2.3.8 *The* linear Poincaré flow *of X is defined over $N_{R(X)}$ by*

$$P_x^t : N_x \to N_{\varphi^t(x)}$$

$$u \mapsto \Pi_{\varphi^t(x)} \circ D\varphi^t(x)u,$$

where $\Pi_x : T_x M \to N_x$ denotes the orthogonal projection on the normal subbundle.

The cocycle condition of the linear Poincaré flow follows from the chain rule.

The linear Poincaré flow is commonly used in the study of flows local behavior; the reason is given by the next proposition.

Proposition 2.3.9 *Let $\Sigma_1 \ni p$ and $\Sigma_2 \ni \varphi^t(p)$ be two cross sections, let $\Phi_t : \Sigma_1 \to \Sigma_2$ be the Poincaré map based on the orbit of p, and let $P_p^t : N_p \to N_{\varphi^t(p)}$ be a map from the linear Poincaré flow. Then the following diagram commutes:*

$$\begin{tikzcd} T_p\Sigma_1 \arrow[r, "D\Phi_t(p)"] \arrow[d, "\Pi_p"'] & T_{\varphi^t(p)}\Sigma_2 \arrow[d, "\Pi_{\varphi^t(p)}"] \\ N_p \arrow[r, "P_p^t"'] & N_{\varphi^t(p)} \end{tikzcd}$$

In particular, if $X(p) \perp T_p\Sigma_1$ and $X(\varphi^t(p)) \perp T_{\varphi^t(p)}\Sigma_2$ then

$$D\Phi_t(p) = P_p^t.$$

Proof: Fix $u \in T_p\Sigma_1$. We have $\Phi_t(x) = \varphi^{\tau(x)}(x)$, where τ is the hitting-time. Differentiating, we obtain

$$D\Phi_t(p) \cdot u = D\varphi^t(p) \cdot u + (D\tau(p) \cdot u)X(\varphi^t p).$$

Write $u = \Pi_p(u) + cX(p)$; then

$$D\Phi_t(p) \cdot u = D\varphi^t(p) \circ \Pi_p(u) + (c + D\tau(p) \cdot u)X(\varphi^t p).$$

Since $\Pi_{\varphi^t p}(X(\varphi^t p)) = 0$, we have

$$\begin{aligned} \Pi_{\varphi^t p} \circ D\Phi_t(p) \cdot u &= \Pi_{\varphi^t p} \circ D\varphi^t(p) \circ \Pi_p(u) \\ &= P_p^t \circ \Pi_p(u), \end{aligned}$$

as we wanted to show. ■

We will often deal with the linear Poincaré flow based on a segment of the orbit of a point p. In this case we will use the following notation:

$$\begin{array}{rccl} P_p^{s,t}: & N_{\varphi^s(p)} & \to & N_{\varphi^t(p)} \\ & u & \mapsto & \Pi_{\varphi^t(p)} \circ D\varphi^{t-s}(\varphi^s(p))u. \end{array}$$

In this notation we include the possibility of $t < s$. So, as a consequence of the cocycle condition, we obtain $(P_p^{t,s})^{-1} = P_p^{s,t}$. In Section 5, the initial base-point will be $0 \in \mathbb{R}^{d-1}$, so we omit it from the notation, yielding $P_s^t = P_0^{s,t}$.

An example of a natural and useful non-linear cocycle appears in the next subsection.

2.4 The orthonormal frame flow

In Section 4, we will define a tubular chart with some useful geometrical properties. To construct this chart, a bundle structure is necessary – the orthonormal frame bundle. We will also need to define a special cocycle over this bundle – the orthonormal frame flow.

Recall that M is a smooth (C^∞) compact manifold of dimension d, endowed with a Riemannian metric. For each $x \in M$, let $\mathfrak{F}_x$ be the set of *orthonormal frames* on the tangent space T_xM (i.e. ordered orthonormal bases of T_xM). Let $\mathfrak{F} = \bigsqcup_{x \in M} \mathfrak{F}_x$. One can define a smooth differentiable structure on $\mathfrak{F}$ so that the obvious projection $\Pi : \mathfrak{F} \to M$ is smooth and defines a fiber bundle, whose fibers are diffeomorphic to the orthonormal group $\mathrm{O}(d)$. This is called the *orthonormal frame bundle* of M.

There is an equivalent way of constructing this bundle: An *oriented flag* at the point $x \in M$ is a nested sequence $F_1 \subset F_2 \subset \cdots \subset F_d$ of vector subspaces of T_xM with $\dim F_i = i$. Given such an oriented flag, there exists an orthonormal frame $(e_1, \ldots, e_d)$ such that F_i is spanned by $e_1, \ldots, e_i$. This correspondence is one-to-one and onto. Therefore $\mathfrak{F}$ can also be viewed as a bundle of oriented flags.

Next, fix a vector field $X \in \mathfrak{X}^r(M)$, and let $\{\varphi^t\}_t$ be the induced flow on M. Then we define a flow on $\mathfrak{F}$ as follows: For each $t \in \mathbb{R}$, the t-image of the orthonormal frame $(e_1, \ldots, e_d) \in \mathfrak{F}_x$ is obtained by applying the Gram–Schmidt process to the frame $(D\varphi^t(x) \cdot e_1, \ldots, D\varphi^t(x) \cdot e_d)$. This is called the *orthonormal frame flow*. It is a flow of class C^{r-1}.

Using the identification between orthonormal frames and oriented flags, the orthonormal frame flow can be described as follows: for each $t \in \mathbb{R}$, the t-image of the flag $F_1 \subset F_2 \subset \cdots \subset F_d = T_xM$ is the flag $D\varphi^t(x)(F_1) \subset D\varphi^t(x)(F_2) \subset \cdots \subset D\varphi^t(x)(F_d)$, where each space is endowed with the induced orientation.

Remark 2.4.1 *More generally, given any vector bundle endowed with a Riemannian metric, one can define an associated orthonormal frame bundle, and given a linear cocycle on the vector bundle, one can define an associated orthonormal frame flow. We will not need those more general constructions.*

2.5 Basic facts about volume crushing

There is a somewhat philosophical obstacle in trying to prove, in a direct way, a theorem of nonexistence. In order to circumvent such issue we present in this section a lemma that reduces our problem to an existence one. It is merely a version for flows of [AB1, Lemma 1].

Lemma 2.5.1 (Criterion for non-existence of acip) *A flow $\{\varphi^t\}$ generated by a vector field $X \in \mathfrak{X}^1(M)$ has no acip iff for every $\epsilon > 0$ there exists a Borel set $K \subset M$ and $T \in \mathbb{R}$ such that*

$$m(K) > 1 - \epsilon \quad \textit{and} \quad m(\varphi^T(K)) < \epsilon.$$

Proof: Notice that the validity of the lemma is unchanged if we replace "$T \in \mathbb{R}$" by "$T \in \mathbb{R}_+$" (just replace K by $M \setminus K$), or by "$T \in \mathbb{N}$" (because the flow up to time 1 cannot distort volumes by more than some constant factor).

We will derive the lemma for the discrete-time version ([AB1, Lemma 1]), which says that a C^1 map $f : M \to M$ has no acip iff for every $\epsilon > 0$ there exists a compact set $K \subset M$ and $T \in \mathbb{N}$ such that

$$m(K) > 1 - \epsilon \quad \text{and} \quad m(f^T(K)) < \epsilon.$$

(Compactness is useful to guarantee measurability of $f^T(K)$ even when f is not invertible.) Notice that if we assume that f is a diffeomorphism, then using the regularity of the measure m, we can replace "compact set" by "Borel set" above.

Notice that a flow $\{\varphi^t\}$ has an acip iff its time-one map φ^1 has an acip; indeed, if μ is an acip for φ^1 then $\bar{\mu} = \int_0^1 \varphi^t_* \mu \, dt$ is an acip for the flow. Hence the lemma follows. ■

For some trivial parts of the dynamics, the crushing property is automatic; for example:

Remark 2.5.2 *Let $X \in \mathfrak{X}^1(M)$. Let M_S be the union all stable manifolds of (hyperbolic) sinks and unstable manifolds of (hyperbolic) sources. If M_S is non-empty then for all $\epsilon > 0$, there is a Borel set $K \subset M_S$ and $T > 0$ such that*

$$m(K) > m(M_S) - \epsilon \qquad \textit{and} \qquad m(\varphi^t(K)) < \epsilon \textit{ for all } t > T.$$

Proof: Take a small neighborhood V_1 (resp. V_2) of the set of sinks (resp. sources), choose T large, and define $K = \varphi^{-T}(V_1) \cup \varphi^T(M \setminus V_2)$. ■

Later on, our perturbations will be supported on the complement of M_S, because in M_S there is nothing to do.

Remark 2.5.3 *One could improve Remark 2.5.2 by including in M_S also the stable (resp. unstable) sets of the hyperbolic attracting (resp. repelling) periodic orbits of X. For example, if the flow is Morse–Smale then the enlarged M_S has full Lebesgue measure, and it follows from Lemma 2.5.1 that there is no acip. (Of course, this also follows directly from the Poincaré Recurrence Theorem and the fact that the recurrent set for a Morse–Smale flow consists of a finite number of periodic orbits.)*

As explained in the Introduction, the following property is essential to our strategy:

Remark 2.5.4 *For each $\epsilon > 0$, the set*

$$\mathcal{V}_\epsilon = \Big\{X \in \mathfrak{X}^1(M) : \textit{there exist a Borel set } K \subset M \textit{ and } T \in \mathbb{R} \textit{ such that} \\ m(K) > 1 - \epsilon \ \textit{ and } \ m(\varphi_X^T(K)) < \epsilon\Big\}$$

is open in the C^1 topology.

Proof: Let $X \in \mathcal{V}_\epsilon$. Take a Borel set $K \subset M$ and $T \in \mathbb{R}$ such that $m(K) > 1 - \epsilon$ and $m(\varphi_X^T(K)) < \epsilon$. Choose a positive $\gamma < \epsilon - m(\varphi_X^T(K))$. Take $Y \in \mathfrak{X}^1(M)$ sufficiently C^1-close to X such that

$$|\det(D\varphi_X^T(p)) - \det(D\varphi_Y^T(p))| < \frac{\gamma}{m(K)},$$

for all $p \in M$. Then we obtain that

$$\begin{aligned} m(\varphi_Y^T(K)) &= \int_K |\det(D\varphi_Y^T(p))| dm(p) \\ &< \int_K \left(|\det(D\varphi_X^T(p))| + \frac{\gamma}{m(K)}\right) dm(p) \\ &= m(\varphi_X^T(K)) + \gamma < \epsilon. \end{aligned}$$

And we conclude that $Y \in \mathcal{V}_\epsilon$. ■

Lemma 2.5.1 and Remark 2.5.4 together imply that that the non-existence of acip is a G_δ property.

2.6 Functions with bounded logarithmic derivative

Recall that the logarithmic derivative of a positive function $f(s)$ is $(\log(f(s)))' = f'(s)/f(s)$.

A simple consequence of the boundedness of the logarithmic derivative is that, in this case, the function presents sub-exponential growth.

Remark 2.6.1 (Sub-exponential growth) *Let $b > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a positive function such that*

$$\left| \frac{d}{ds}(\log(f(s))) \right| < b, \;\; \forall s \in \mathbb{R}.$$

Then

$$e^{-b|s|} < f(s) < e^{b|s|}, \quad \forall s \in \mathbb{R}.$$

Let $I = [\alpha, \beta] \subset \mathbb{R}$ be a compact interval and let $a > \beta - \alpha$. We will use the following notation:

$$I_a \equiv [\alpha + a, \beta - a] \qquad \text{and} \qquad I^a \equiv [\alpha - a, \beta + a].$$

Proposition 2.6.2 *Given $b > 0$, $t_0 > 0$ and $\gamma \in (0, 1)$, there exists $a_0 > 0$ such that for all $0 < a < a_0$, for any interval I with $|I| > t_0$ and for all positive $f \in C^1(\mathbb{R}, \mathbb{R})$ such that*

$$|f'(s)| \leq bf(s), \quad \forall s \in \mathbb{R}$$

the following holds:

$$\int_{I_a} f(s)ds > (1 - \gamma) \int_{I^a} f(s)ds.$$

Before proving this proposition, we need a lemma:

Lemma 2.6.3 *Let f and b be as in the previous Proposition. Then given $\alpha < \beta$, we have that*

$$b^{-1} \max\{f(\alpha), f(\beta)\}(1 - e^{-b(\beta-\alpha)}) < \int_{\alpha}^{\beta} f(s)ds < b^{-1} \min\{f(\alpha), f(\beta)\}(e^{b(\beta-\alpha)} - 1).$$

Proof: Let $\alpha < t < \beta$. By the hypothesis' inequality,

$$b > \left| \frac{f'(s)}{f(s)} \right|, \tag{2.12}$$

for all $s \in \mathbb{R}$. Integrating both sides from α to t, we obtain

$$\begin{aligned} b(t-\alpha) &> \int_\alpha^t \left|\frac{f'(s)}{f(s)}\right| ds \\ &> \left|\int_\alpha^t \frac{f'(s)}{f(s)} ds\right| \\ &= \left|\log\left(\frac{f(t)}{f(\alpha)}\right)\right|. \end{aligned}$$

Which leads us to

$$f(\alpha)e^{-b(t-\alpha)} < f(t) < f(\alpha)e^{b(t-\alpha)}. \tag{2.13}$$

If we integrate both sides of (2.12) from t to β we will obtain a similar conclusion:

$$f(\beta)e^{-b(\beta-t)} < f(t) < f(\beta)e^{b(\beta-t)}. \tag{2.14}$$

Using the righthand side of both (2.13) and (2.14) we conclude that

$$\int_\alpha^\beta f(t)dt < b^{-1}\min\{f(\alpha), f(\beta)\}(e^{b(\beta-\alpha)} - 1).$$

The same way, using the lefthand side of (2.13) and (2.14) we get

$$\int_\alpha^\beta f(t)dt > b^{-1}\max\{f(\alpha), f(\beta)\}(1 - e^{-b(\beta-\alpha)}).$$

■

Proof of Proposition 2.6.2: Let

$$a_0 = \min\left\{\frac{t_0}{2}, (2b)^{-1}\log\left(\gamma\frac{(1-e^{-bt_0})}{2} + 1\right)\right\}$$

and assume that $I = [\alpha, \beta]$, with $|\beta - \alpha| < t_0$. Take $0 < a < a_0$ and $T > t_0$ and denote

$$A = \int_{\alpha-a}^{\beta+a} f(s)ds.$$

Our goal is to prove that

$$\int_{\alpha-a}^{\alpha+a} f(s)ds < (\gamma/2)A \quad \text{and} \quad \int_{\beta-a}^{\beta+a} f(s)ds < (\gamma/2)A.$$

Since the proofs of both inequalities are totally analogous, we present only the the proof of the first one.

From Proposition 2.6.2 and the fact that $|\beta - \alpha| < t_0$ we have that

$$
\begin{aligned}
A &\geq b^{-1} \max\{f(\alpha - a), f(\beta + a)\}(1 - e^{-b(t_0+2a)}) \\
&\geq b^{-1} f(\alpha - a)(1 - e^{-b(t_0+2a)}). \qquad (2.15)
\end{aligned}
$$

Again from Proposition 2.6.2, we obtain

$$
\begin{aligned}
\int_{\alpha-a}^{\alpha+a} f(s)ds &\leq b^{-1} \min\{f(\alpha - a), f(\alpha + a)\}(e^{2ba} - 1) \\
&\leq b^{-1} f(\alpha - a)(e^{2ba} - 1). \qquad (2.16)
\end{aligned}
$$

From Inequalities (2.15), (2.16) and the fact that $0 < a < a_0$, we conclude that

$$
\begin{aligned}
\int_{\alpha-a}^{\alpha+a} f(s)ds &\leq \frac{A(e^{2ba} - 1)}{1 - e^{-b(t_0+2a)}} \\
&< \frac{A(e^{2ba} - 1)}{1 - e^{-bt_0}} \\
&< A\frac{\exp(\log(\frac{\gamma(1-e^{-bt_0})}{2} + 1)) - 1}{1 - e^{-bt_0}} \\
&= A\frac{\frac{\gamma(1-e^{-bt_0})}{2}}{1 - e^{-bt_0}} \\
&= A\frac{\gamma}{2}.
\end{aligned}
$$

■

2.7 Vitali Covering

In this work, we use a version of the Vitali Covering Theorem (usually stated in $\mathbb{R}^d$) for compact Riemannian manifolds and include the possibility of the sets in the covering not being balls for the Riemannian metric. For the Theorem still hold in this more general setting, we need that the sets in the cover satisfy a *roundness* property. Roughly speaking, this property means that the sets can be sandwiched by balls for which the ratio between the radii is uniformly bounded. This property is defined in [P, Appendix E].

Definition 2.7.1 (Quasi-roundness) *Let M be a Riemannian Manifold and $x \in M$. We say that $U \subset M$ is a K-quasi-round neighborhood of x if there* *exists $r > 0$ (lower then the injective radius) such that*

$$B_{K^{-1}r}(x) \subset U \subset B_r(x),$$

where $B_r(x)$ is the Riemannian ball around x with radius r.

Definition 2.7.2 (Vitali Cover) *Let $S \subset M$ and let $K > 1$. If $\mathcal{V} = \{V_\alpha\}$ is a cover of S such that for m-a.e. $x \in S$ and for all $r \in (0, \sup_\alpha \operatorname{diam}(U_\alpha))$ there exists a K-quasi-round neighborhood $U \subset \mathcal{V}$ of x with $U \subset B_r(x)$, then we say that $\mathcal{V}$ is a* Vitali Cover of S*.*

Theorem 2.7.3 (Vitali Covering Theorem) *If $\mathcal{V}$ is a Vitali cover of S, then there exists a countable pairwise disjoint family $\{V_j\}_j \subset \mathcal{V}$ such that*

$$m\left(S \backslash \bigcup_j V_j\right) = 0.$$

Proposition 2.7.4 *Let M and N be compact Riemannian manifolds and let $F : U \subset M \to F(U) \subset N$ be a diffeomorphism with uniform bounded non-conformality, that is, there exists $C > 1$ such that*

$$NC(DF(p)) < C, \quad \forall p \in U.$$

Then for all $x \in U$, there exists $r > 0$ such that for all K-quasi-round neighborhood $V \ni x$ with $\operatorname{diam}(V) < r$, $F(V)$ is a KC-quasi-round neighborhood of $F(x)$.

Proof: Since we can take V arbitrarily small, the proof follows from Proposition 2.2.1. ■

Remark 2.7.5 *We conclude, by the previous Proposition and Theorem 2.7.3, that Vitali Covers are preserved by diffeomorphisms with uniform bounded non-conformality.*