6
Global Crushing

Throughout the text we presented several ingredients necessary for the proof of Theorem 1.2.1. This section is devoted to combine all these pieces together. Before starting the proof, we will state a slightly different version of the "non-invariant Rokhlin lemma" from [AB1], which will be used in the scope of the proof.

Lemma 6.0.2 (Avila, Bochi) *Let $(\Lambda, \mathcal{K}, \sigma)$ be a Lebesgue space. Assume that $f : K \to K$ is a bi-measurable map which is continuous on an open subset with total measure and which is non-singular with respect to σ. Assume also that $\sigma(P_f) = 0$, where P_f is the set of periodic points for f. Then given any $\epsilon_0 > 0$ and $n, k \in \mathbb{N}$, with $k \leq n$, there exists an open set $B \subset \Lambda$ such that $f^{-i}(\bar{B}) \cap \bar{B} = \emptyset$ for $1 \leq i < n$,*

$$\sum_{i=0}^{n-1} \sigma(f^i(U)) > 1 - \epsilon_0 \quad \textit{and} \quad \sum_{i=n-k-1}^{n-1} m(f^i(B)) < \frac{k}{n} + \epsilon_0.$$

Proof: Lemma 6.0.2 follows immediately from Theorem 2 and Remark 1 from [AB1] (Actually the result from [AB1] is more general because it deals with non-necessarily invertible maps). ∎

By Lemma 2.5.1 and Remark 2.5.4, we only need to show that, for any $0 < \delta < 1$, the set $\mathcal{V}_\delta$ is dense in the C^1-topology. So we fix $0 < \delta < 1$ and $X \in \mathfrak{X}^1(M)$ and explain how to construct a perturbation of X in $\mathcal{V}_\delta$. By Proposition 2.1.10, we can assume, without loss of generality, that X is a C^3 vector field with only hyperbolic *periodic orbits* (in particular the set of periodic orbits is finite).

For a fixed $\epsilon > 0$, we are going to show how to find $Y \in \mathcal{V}_\delta$ ϵ-close to X in the C^1 topology.

First, let $\Sigma \subset M$ be the transverse section given by Lemma 3.0.6 and (Λ, f, σ) its discrete induced dynamics. Take $t_0 = t_0(\epsilon, \delta) > 0$ as in Lemma 5.0.22. We will denote by $\delta_j = \delta_j(\delta) > 0$ the positive constant given by Remark

3.0.10, with $j \in \mathbb{N}$ in the role of n, δ_j in the role of $\delta > 0$ and with $\delta/5$ in the role of $\epsilon > 0$. That is,

$$\sigma(A) < \delta_j \Rightarrow m(\mathcal{T}_j(A)) < \frac{\delta}{5}. \tag{6.1}$$

Recall that there is a full-measure open subset G of the transverse section Σ where the first-return map f and the return-time function τ are continuous. Let G_n be the set of points in G that have n well-defined first-returns in G. Of course, $G_n \supset \Lambda$.

Let $\tau^- = \min_{x\in\Lambda}\{\tau(x)\}$, where $\tau : \Lambda \to \mathbb{R}_+$ is the return time function that defines f, and define

$$k = \left\lceil \frac{t_0}{\tau^-} \right\rceil \qquad \text{and} \qquad n = \left\lceil \frac{k}{\delta_1} \right\rceil .$$

Take $B \subset \Lambda$ the base of the tower given by Lemma 6.0.2 with respect to $k, n \in \mathbb{N}$ and $\delta_1/2$ (in the place of ϵ_0). So B is a relatively open set of Λ.

Define

$$T(x) = \sum_{j=0}^{n-1} \tau(f^j(x))$$

and notice that $T(x) > t_0$ for each $x \in B$. Also observe that, since $\sigma(G_n) < \infty$, there exists $T_0 > 0$ such that

$$A_{T_0} \equiv \{x \in G_n : T(x) < T_0\}$$

satisfies

$$\sigma(B \backslash A_{T_0}) < \frac{\delta_n}{2}. \tag{6.2}$$

Let $\kappa = \kappa(T_0)$ be given by Lemma 5.0.22. Consider the family $\mathcal{R}$ of all κ-rectangles R whose center is a non-periodic point $p \in B \cap A_{T_0}$, have diameter less than the number $\rho = \rho(p)$ provided by Lemma 5.0.22, and are contained in G_n (which is an open subset of the transverse section).

For each rectangle R in the family $\mathcal{R}$, we can apply Lemma 5.0.22 with $T = T(p) \in (t_0, T_0)$ (where p is the center of the rectangle) and obtain an ϵ-perturbation of X supported on the tube

$$U = U(R) = \bigcup_{t\in[0,T(p)]} \varphi^t(R).$$

For later use, let us compare the tube U with the tube

$$\mathcal{T}_n(R) = \bigcup_{x\in R} \bigcup_{t\in[0,T(x)]} \varphi^t(x).$$

Recall that R is contained in the set G_n where the function T is continuous. Therefore, if the diameter of R is small enough then $\mathcal{T}_n(R)$ and $U(R)$ are close in the following sense:

$$\frac{m(U(R)\triangle\mathcal{T}_n(R))}{m(U(R))} < \frac{\delta}{5}. \tag{6.3}$$

Reducing the family $\mathcal{R}$ if necessary, we assume that this property above holds for every $R \in \mathcal{R}$.

Now, since $\mathcal{R}$ is a Vitali cover of $B \cap A_{T_0}$ (Remark 4.0.21), there exists a finite number J of pairwise disjoint κ-rectangles R_j with diameter smaller than ρ_0, centered in non-periodic points p_j and such that

$$\sigma\left((B \cap A_{T_0})\setminus\bigcup_{j=1}^{J} R_j\right) < \frac{\delta_n}{2}. \tag{6.4}$$

Denote $U_j = U(R_j)$, the support of the j-th perturbation of X. By construction, the sets U_j have disjoint closures. Therefore, we can paste together the J perturbations and find a single ϵ-perturbation of X whose restriction to each U_j has the properties provide by Lemma 5.0.22, namely: There exists

$$V_j \subset U_j^- = \bigcup_{t\in[0,T-t_0]} \varphi^t(R_j)$$

such that

1. $\dfrac{m(V_j)}{m(U_j^-)} > 1-\delta;$

2. $\varphi^t_X(\overline{V_j}) \subset U_j \qquad \forall t \in [0,t_0];$

3. $\varphi^t_{\widetilde{X}}(\overline{V_j}) \subset U_j \qquad \forall t \in [0,t_0];$

4. $\dfrac{m(\varphi^{t_0}_{\widetilde{X}}(V_j)\triangle U_j^+)}{m(U_j^+)} < \delta.$

The compact set to be crushed is $K = \bigsqcup_j V_j$ and the perturbation is $\widetilde{X}$. Now we need to verify if K and $\widetilde{X}$ satisfy the crushing property, that is, if:

1. $m(M_R \cap K) > 1-\delta;$

2. $m(\varphi^{t_0}_{\widetilde{X}}(K)) < \delta.$

The second inequality is a direct consequence of Lemma 5.0.22. In fact, since $\varphi^{t_0}_{\widetilde{X}}(V_j) \subset U_j^+$, for all j and since the U_j's are pairwise disjoint, then

$$\varphi^{t_0}_{\widetilde{X}}\left(\bigsqcup_j V_j\right) = \bigsqcup_j \varphi^{t_0}_{\widetilde{X}}(V_j)$$

and, therefore,

$$m\left(\bigsqcup_j \varphi_{\widetilde{X}}^{t_0}(V_j)\right) \leq \frac{\delta}{5}\cdot m\left(\bigsqcup_j U_j^+\right) \leq \delta.$$

The first inequality will be verified in 4 steps:

1. $m\left(M_R\backslash \mathcal{T}_n(B)\right) < \delta/5$;
2. $m\left(\mathcal{T}_n(B)\backslash \bigsqcup_j U_j\right) < 2\delta/5$;
3. $m\left(\bigsqcup_j U_j\backslash \bigsqcup_j U_j^-\right) < \delta/5$;
4. $m\left(\bigsqcup_j U_j^-\backslash \bigsqcup_j V_j\right) < \delta/5$.

In order to verify (1), note that, by Lemma 6.0.2,

$$\sigma\left(\Lambda\backslash \bigcup_{i=0}^{n-1} f^i(B)\right) < \delta_1$$

and that, since

$$\mathcal{T}_1\left(\Lambda\backslash \bigcup_{i=0}^{n-1} f^i(B)\right) = M_R\backslash \mathcal{T}_n(B),$$

we have, by property (6.1), that $m(M_R\backslash \mathcal{T}_n(B)) < \frac{\delta}{5}$.

To verify the second inequality, first observe that $\bigsqcup_j R_j \subset B$ and, therefore,

$$\sigma\left(B\backslash \bigsqcup_j R_j\right) \leq \sigma(B\cap A_{T_0}^c) + \sigma\left((B\cap A_{T_0})\backslash \bigsqcup_j R_j\right).$$

From inequalities (6.2) and (6.4), we obtain

$$\sigma\left(B\backslash \bigsqcup_j R_j\right) \leq \delta_n.$$

So we can apply Equation (6.1) and conclude that

$$m\left(\mathcal{T}_n(B)\backslash \bigsqcup_j \mathcal{T}_n(R_j)\right) < \frac{\delta}{5}.$$

From Equation (6.3), we have that

$$m\left(\bigsqcup_j \mathcal{T}_n(R_j)\backslash \bigsqcup_j U_j\right) < \frac{\delta}{5}$$

and we conclude that

$$m\left(\mathcal{T}_n(B)\backslash\bigsqcup_j U_j\right) = m\left(\mathcal{T}_n(B)\backslash\bigsqcup_j \mathcal{T}_n(R_j)\right) + m\left(\bigsqcup_j \mathcal{T}_n(R_j)\backslash\bigsqcup_j U_j\right)$$
$$\leq \frac{\delta}{5} + \frac{\delta}{5} = \frac{2\delta}{5}.$$

The third one follows from Lemma 6.0.2. We only need to show that

$$U_j\backslash U_j^- \subset \mathcal{T}_1\left(\bigcup_{i=n-k-1}^{n-1} f^i(R_j)\right),$$

but this follows from the fact that we chose k such that $\varphi^{t_0}(p)$ hits B no more then k times, for all $p \in B$. Therefore, by Lemma 6.0.2,

$$\sigma\left(\bigcup_{i=n-k-1}^{n-1} f^i(R_j)\right) < \delta_1$$

and, then, by Equation (6.1),

$$m\left(\bigsqcup_j (U_j\backslash U_j^-)\right) < \frac{\delta}{5}.$$

Finally, the fourth inequality is a direct consequence of the Main Lemma (5.0.22) and the fact that the sets U_j are pairwise disjoint.

Without loss of generality, we could consider t_0 greater then the δ-crushing time for $M\backslash M_R$ (See Remark 2.5.2) and then, the δ-crushing property would be seen in the whole manifold M.