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Introduction

We consider a Riemannian compact manifold M and the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms from M to itself, endowed with the usual uniform C^1-topology.

We aim to understand dynamical properties that present some "persistence". More precisely, we look for those properties that are verified in a locally dense subset of $\mathrm{Diff}^1(M)$. However, since the intersection of two dense subsets is not necessarily dense, a more useful concept is the one of *locally residual subset*. Given an open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$, we say that $\mathcal{R}$ is a *residual* subset of $\mathcal{U}$ if it is a countable intersection of open and dense subsets of $\mathcal{U}$. Locally generic subsets are also locally dense. The importance of generic properties relies in the fact that, given two residual subsets $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{U}$, the intersection $\mathcal{R}_1 \cap \mathcal{R}_2$ is still a residual subset of $\mathcal{U}$. Then, we can gather in a unique residual subset of $\mathcal{U}$ a countable many amount of properties that hold residually in $\mathcal{U}$. A property of a diffeomorphism f is *locally generic* if it holds in a residual subset of a C^1-open neighborhood of f.

A stronger form of persistence is *robustness.* A property of a diffeomorphism f is *robust* if it holds in a C^1-open neighborhood of f. Two important robust properties are *hyperbolicity* and *partial hyperbolicity.* Nowadays hyperbolic systems are fairly well understood from the topological and ergodic points of view. Although partially hyperbolic systems share some important features with hyperbolic ones (for instance, there are uniformly contrating and expanding directions and invariant foliations associated to them) our knowlodge of them is still incipient (although relevant progress in this subject were done recently). Naively, when studying partially hyperbolic systems, a natural problem is to verify what kind of hyperbolic behaviours "survive" in this setting.

One of the properties of hyperbolic transitive sets that motivates this thesis is the denseness of the invariant manifolds and its version for partially hyperbolic diffeomorphism (see, Theorem 1.1).

Another motivation are the spectral decomposition theorems of the sets containing the relevant part of the dynamics (limit set, non-wandering set, chain recurrent set). In the C^1-generic case this set splits into pairwise disjoint chain-recurrence classes (Conley decomposition, see (19)), and each class containing a periodic point is a homoclinic class, see (15). Homoclinic classes are a special type of transitive set, see Definition 4.1. In the hyperbolic case, the decomposition consists of finitely many sets, called *basic pieces*, and each set is a homoclinic class, see (31). Specially important sets in this decomposition are the attractors and repellers. These sets persist, are robustly transitive, and their basins form an open and dense subset of the ambient. However, these basins may fail to have full measure (see the example of a horseshoe with positive measure in (16)). We aim to understand the structure of partially hyperbolic attractors. Indeed, we see that some of the features above are still valid for them.

Now we state our results and motivations in a more precise way. Consider a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and an f-invariant set Λ with a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$, where the extremal fibres E^s and E^u are non-trivial and uniformly contracting and uniformly expanding, respectively, by the action of the derivative Df (see Definition 3.1 to a more accurate definition of partially hyperbolic sets). When $\Lambda = M$ we say that f is a *partially hyperbolic diffeomorphism*. In this case, according to (23), the bundles E^s and E^u are integrable to invariant foliations of M that we call, respectively, the *strong stable* and the *strong unstable* foliation. A foliation is *minimal* if the orbit of each leaf is dense in M. When the strong stable (resp. unstable) foliation is minimal we speak of s-minimality (resp. u-minimality).

By a robustly transitive attractor $\Lambda_f(U)$, we mean a compact f-invariant set that is isolated (so it has upper semicontinuations for every diffeomorphism g in a small neighbothood of f, see Remark 2.2), it attracts some neighborhood U of it, and its semicontinuations $\Lambda_g(U)$ are transitive: there exist a point in the $\Lambda_g(U)$ whose forward orbit is dense in $\Lambda_g(U)$. See Section 2 for the precise definitions.

When the attractor coincide with the whole manifold in a robust way, then we say that f is a robustly transitive diffeomorphism.

Let us denote by $\mathrm{RTPH}_1(M)$ the subspace of $\mathrm{Diff}^1(M)$ consisting of robustly transitive partially hyperbolic diffeomorphisms with one-dimensional center bundle.

Theorem 1.1 ((12),(22)) *There is an open and dense subset of* $\mathrm{RTPH}_1(M)$ *consisting of diffeomorphisms which are either robustly s-minimal or robsutly u-minimal.*

When the partial hyperbolicity is defined only in a subset Λ of M, the results on (23) also guarantee the existence of invariant laminations (by an abuse of notation, we also call them foliations) tangent to the strong bundles. To state a result similar to Theorem 1.1 in the setting of proper attractors, we need a suitable notions of s and u-minimality for "set foliations". See Section 5.1 for the precise definitions. The next theorem summarizes the main results we obtain about partially hyperbolic attractors.

Theorem A *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ with the following property: For every diffeomorphism $f \in \mathcal{R}$ and every opens set U, if $\Lambda_f(U)$ is a transitive attractor having a partially hyperbolic splitting with one-dimensional center bundle then there is an open neighborhood $\mathcal{U}$ of f such that:*

Let $\mathcal{G}_s$ (resp. $\mathcal{G}_u$) be the subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms g such that $\Lambda_g(U)$ is s-minimal (resp. u-minimal). Then $\mathcal{G}_s \cup \mathcal{G}_u$ is a residual subset of $\mathcal{U}$.

In addition, if the set $\Lambda_f(U)$ is robustly transitive then $\mathrm{int}(\mathcal{G}_s) \cup \mathrm{int}(\mathcal{G}_u)$ is an open and dense subset of $\mathcal{U}$.

Indeed the previous theorem holds when the open set $\mathcal{U}$ is a compatible neighborhood of f, see Definition 3.11. Very roughly, this means that for any $g \in \mathcal{U}$ the set $\Lambda_g(U)$ shares those robust properties of $\Lambda_f(U)$ as the fact of being an attractor, the existence of a partially hyperbolic splitting and so on.

A first difficulty to adapt the global Theorem 1.1 to our local case is that we must guarantee that some intersections occur inside the attractor. Another difficulty is that the stable leaves may go far from the region where the partial hyperbolicity is defined and do not have a uniform contraction. Finally, a little more technical constraint is that a co-dimension one manifold may not divide locally the attractor into different components. Hence, if one tries to intersect such manifold with a curve starting at some point of the attractor and tangent to the complementary direction, we may have no choice in which side this curve comes.

The u or s-minimality is also a sufficient condition to guarantee that a robustly transitive set (not necessarily an attractor) is robustly inside a homoclinic class.

Theorem B *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ satisfying the following. Let $\Lambda_f(U)$ be a robustly transitive set that is partially hyperbolic with one-dimensional center bundle and u or s-minimal. Then, given any hyperbolic periodic point $p \in \Lambda_f(U)$, there is an open set $\mathcal{W}_p \subset \mathrm{Diff}^1(M)$, with $f \in \overline{\mathcal{W}_p}$, such that $\Lambda_g(U)$ is contained in the homoclinic class of p_g (the continuation of p for g) with respect to U, for all $g \in \mathcal{W}_p$.*

Since the attractors contain its unstable manifolds (and in particular its homoclinic classes), Theorem B implies the following consequence for attractors.

Corollary C *Under the hypotheses of Theorem A, a robustly transitive attractor is, robustly, a homoclinic class.*

Another interesting problem is to find conditions for a transitive attractor to be robustly transitive. As far as we known, it is still an open question if there is a generically transitive attractor that is not robustly transitive. Next theorem gives a sufficient condition to verify robust transitivity.

Theorem D *There is a residual set* $\mathcal{R} \subset \mathrm{Diff}^1(M)$ *satisfying the following. Let* $f \in \mathcal{R}$ *and* $\Lambda_f(U)$ *be a* $(s, 1, u)$*-partially hyperbolic attractor that is both u and s-minimal. Then* $\Lambda_f(U)$ *is robustly u and s-minimal.*

In Theorem 5 in (2), it is stated the following dichotomy for homoclinic classes of C^1-generic diffeomorphisms with non-empty interior: the class is either the whole manifold or it is accumulated by homoclinic classes (note that this last possibility cannot occur for attractors). Actually, under partial hyperbolicity assumptions there are stronger versions of this result. First, when the whole manifold is partially hyperbolic, the accessibility properties (see for example (20)) and the arguments in (17) imply that the class is the whole manifold, and that this property is indeed robust (see Theorem 2 in (2)). Second, when the partial hyperbolicity is local (defined only over the homoclinic class) then the class is the whole manifold and this property is just locally generic (see Corollary 1 in (2)). Here we improve Corollary 1 in (2) in the case of attractors, proving that in this case the fact that the class has empty interior is a robust property. Indeed we have the following general result that does not involve neither accessibility nor genericity.

Theorem E *Every s or u-minimal proper set has empty interior.*

From Theorems A and E we get the following corollary.

Corollary F *Under the hypotheses of Theorem A, if* $\Lambda_f(U)$ *is a robustly transitive proper attractor then it has robustly empty interior.*

For further related results on elementary pieces of dynamics with non-empty interior see the discussion in (2) and also (27) where the bi-Lyapunov stable case was considered.

Another problem concerns the Lebesgue measure of these attractors. In (4) it is proved that if f is C^{1+} and Λ is a hyperbolic proper set for f, then Λ has zero Lebesgue measure. We can also say something in our context about the Lebesgue measure when the attractor is s-minimal. In addiction, in the s-minimal case the property of robust empty interior holds even if the attractor is not robustly (but generically) transitive.

Theorem G *Under notation and hypotheses of Theorem A, there is a open set $\mathcal{U}_s \subset \mathcal{U}$ such that*

- *$\mathcal{G}_s$ is a residual subset of $\mathcal{U}_s$,*
- *$\Lambda_g(U)$ has empty interior for all $g \in \mathcal{U}_s$, and*
- *$\Lambda_g(U)$ has zero Lebesgue measure for all $g \in \mathcal{G}_s$.*

Recently in (3), it is proved a spectral decomposition theorem for a more general class of sets: chain-transitive locally maximal sets. Their theorem holds in a residual subset of $\mathrm{Diff}^1(M)$. Here we prove a spectral decomposition for s and u-minimal homoclinic classes, that do not rely on genericity.

Theorem H *Let $\Lambda = H(p, f)$ be an s-minimal (or u-minimal) (s, c, u)-partially hyperbolic attractor with minimal constant d. Then Λ admits a unique spectral decomposition with exactly d components.*

As a consequence of this theorem and Theorem A, we can obtain a robust spectral decomposition for robustly transitive attractors.

Theorem I *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ satisfying the following. For every $f \in \mathcal{R}$ and $U \subset M$, if $\Lambda_f(U)$ is a partially hyperbolic robustly transitive attractor with one dimensional center bundle, then $\Lambda_f(U)$ has a robust spectral decomposition: every g in a small neighborhood of f has a spectral decomposition whose pieces are the continuations of the pieces of the spectral decomposition of Λ_f.*

This thesis is organized as follows. In Section 2 we briefly define the basic objects and properties we are interested to study. Section 3 describes the partially hyperbolic structure in the tangent space, the foliations that are derived from it and their elementary properties. In Section 4 we overview the main tools of C^1-generic dynamics. We mainly focus on properties of homoclinic classes and generically transitive sets. We define the main concept of minimality of foliations for these sets in Section 5. Theorem D, Theorem E, Corollary F, and Theorem B correspond, respectively, to Theorem 5.18, Theorem 5.2, Corollary 5.3 and Theorem 5.9. Subsection 5.4 treat the special case of s-minimal attractors, proving Theorem G. In Section 6 we prove Theorem A and Corollary C that correspond, respectively, to Theorem 6.1 and Corollary 6.3. The proof of Theorem A is the content of Subsection 6.2. Finally, Section 7 study the spectral decomposition of these attractors, proving Theorems H and I.