3
Invariant foliations (for sets)

This thesis deals with invariant compact sets admitting a *partially hyperbolic* splitting on its tangent bundle (see definition 3.1). This condition leads to the existence of dynamically defined immersed submanifolds through each point in the set, see Proposition 3.4. The study of the dynamical properties of these leaves is one of the subjects of this thesis.

In what follows, we denote by $||.||$ the Riemannian metric on M.

Definition 3.1 (partially hyperbolic set) A compact invariant set Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *partially hyperbolic* if there is a splitting of its tangent bundle $T_\Lambda M = E^s \oplus E^c \oplus E^u$ satisfying:

- at least one of the bundles E^s and E^u is non trivial,
- the bundles E^s, E^c, and E^u are Df-invariant. That is,

$$Df_x(E^i(x)) = E^i(f(x)) \quad \text{for every } x \in \Lambda \text{ and } \; i \in \{s, c, u\},$$

- the bundles E^s and E^u are, respectively, uniformly contracting and uniformly expanding. That is, there are constants $C > 0$ and $0 < \lambda < 1$ such that for every $x \in \Lambda$ and every unitary vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$ it holds that

$$||Df_x^n.v^s|| < C\lambda^n \quad \text{and} \quad ||Df_x^{-n}.v^u|| < C\lambda^n$$

  for every $n \in \mathbb{N}$, and
- intermediate behavior of the central bundle (domination): there is $n_0 \in \mathbb{N}$ such that, for every $x \in \Lambda$ and every unitary vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, and $v^u \in E^u(x)$, it holds that

$$||Df_x^{n_0} v^s|| < ||Df_x^{n_0} v^c|| < ||Df_x^{n_0} v^u||$$

The bundles E^s, E^c, and E^u are called, respectively, *strong stable*, *central*, and *strong unstable bundles*. When both bundles E^s and E^u are non trivial, we say that the set Λ is *strongly partially hyperbolic*.

In (21) it is proved that there exists a metric on M (called *adapted metric*) that is equivalent to $||.||$ and such that $C = 1$. In the rest of this thesis, we always assume that the metric on M is adapted.

Partial hyperbolicity is a robust property. This means that if Λ is a partially hyperbolic set then there are neighborhoods U of Λ and $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that, for all $g \in \mathcal{U}$, any compact g-invariant set $K \subset U$ is partially hyperbolic with the same kind of splitting (that is, with the same dimensions of the bundles E^s, E^u and E^c). We refer to Appendix B of (13) for a list of elementary properties of partially hyperbolic sets.

In what follows we assume that the dimensions of $E^s(x)$, $E^c(x)$, and $E^u(x)$ do not depend on the point $x \in \Lambda$. Due to the Df-invariance, this assumption is automatically satisfied when Λ is transitive. We denote them by $s = \dim(E^s)$, $c = \dim(E^c)$, and $u = \dim(E^u)$.

If $T_\Lambda M$ admits a spitting where the central bundle is trivial ($c = 0$), then Λ is *hyperbolic*. It is a well known fact that hyperbolic sets persist under small perturbations and depend continuously on the diffeomorphism. In particular, given a hyperbolic periodic point p of a C^1-diffeomorphism f and g in a small neighborhood of f, there is a continuation of p for g, that we denote by p_g. The continuation p_g has the same period of p, its orbit is close to the orbit of p, and its stable manifold has the same dimension of the stable manifold of p (which we call *index* of p and denote by $\mathrm{index}(p)$).

In the rest of this thesis we always assume that, for a strongly partially hyperbolic set Λ, the constant c is positive and E^c is neither uniformly contracting nor uniformly expanding. In this case, we say that Λ is a (s, c, u)-*partially hyperbolic* set.

Definition 3.2 Let $\Lambda \subset M$ be a compact subset of the manifold M. A *foliation* $\mathcal{F}$ of Λ is a family of immersed submanifolds of M of the same dimension (called *leaves*), satisfying the following:

1. each leaf of the foliation intersects Λ, and every point of Λ belongs to some leaf,
2. the leaves are pairwise disjoint, and
3. the leaves of $\mathcal{F}$ vary continuously on $x \in \Lambda$.

Denote by $\mathcal{F}(x)$ the leaf of $\mathcal{F}$ that contains the point $x \in \Lambda$. The continuous dependence of the leaves of $\mathcal{F}$ means the following. For every $x \in \Lambda$, there exist a neighborhood $V_x \subset \Lambda$ of x, a disk $W_x \subset \mathcal{F}(x)$ centered at x and with same dimension as $\mathcal{F}(x)$, and a continuous map $\phi_x : V_x \to Emb(W_x, M)$ such that $\phi_x(x)$ is the inclusion of W_x in M and $\phi_x(z)(W_x)$ is a neighborhood of z inside $\mathcal{F}(z)$. Here $Emb(N, M)$ denotes the space of all embeddings from the manifold N to M.

Given $f \in \text{Diff}^1(M)$ and an f-invariant set Λ, the foliation $\mathcal{F}$ is said to be *f-invariant* if $f(\mathcal{F}(x)) = \mathcal{F}(f(x))$ for every $x \in \Lambda$.

Definition 3.3 The *saturation* of a set K by a foliation $\mathcal{F}$ is the set consisting of the union of all leaves of points in K. A set K is *saturated* by $\mathcal{F}$ if the saturation of K equals K (i.e, for every $x \in K$ we have $\mathcal{F}(x) \subset K$).

The following result summarizes some well known properties of strongly partially hyperbolic sets (see, for instance, Section 6.2 of (29)).

Proposition 3.4 *Let Λ be a (s, c, u)-partially hyperbolic set of a diffeomorphism $f \in \text{Diff}^r(M)$. Then there are f-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ of Λ satisfying the following properties:*

1. *The leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ are C^r immersed submanifolds of M of dimensions s and u, respectively, called* strong stable *and* strong unstable *leaves.*

2. *The strong stable and strong unstable leaves are tangent, respectively, to the strong stable and strong unstable bundles. That is, for every $x \in \Lambda$, it holds that $T_x\mathcal{F}^s(x) = E^s(x)$ and $T_x\mathcal{F}^u(x) = E^u(x)$.*

3. *The diffeomorphism f exponentially contracts the leaves of $\mathcal{F}^s$, and its inverse f^{-1} exponentially contracts the leaves of $\mathcal{F}^u$. This means that there is $0 < \lambda < 1$ such that, for every $x \in \Lambda$, $z_1, z_2 \in \mathcal{F}^s(x)$ and $w_1, w_2 \in \mathcal{F}^u(x)$, there are $C_1 > 0$ and $C_2 > 0$ satisfying the following:*

$$d(f^n(z_1), f^n(z_2)) < C_1\lambda^n \text{ and } d(f^{-n}(w_1), f^{-n}(w_2)) < C_2\lambda^n$$

   for all $n \geq 1$. Here d denotes the distance induced on the strong stable (or unstable) leaf by the Riemannian metric of M.

In general, the constant C_1 (resp. C_2) depends on the points z_1 and z_2 (resp. w_1 and w_2). However, if the whole manifold M is partially hyperbolic, then we can take these constants to be uniform.

Remark 3.5 Let Λ be a (s,c,u)-partially hyperbolic set. For every hyperbolic periodic point $p \in \Lambda$, the set $\mathcal{F}^s(p)$ is a subset of its stable manifold $W^s(p)$, thus $\operatorname{index}(p) \geq s$. Analogously, the strong unstable leaf of p is a subset of its unstable manifold $W^u(p)$. Thus, $s+c+u-\operatorname{index}(p) \geq u$. In particular, when the central bundle is one dimensional ($c=1$), the index of a hyperbolic periodic point p is either s or $s+1$.

Fixed $r>0$, we denote by $\mathcal{F}^s_r(x)$ the open ball of radius r centered at x, relative to the induced distance on $\mathcal{F}^s(x)$. Given a hyperbolic periodic point p, we define the local manifolds $W^s_r(p)$ and $W^u_r(p)$ in the same way.

Lemma 3.6 For every $r>0$ sufficiently small and $x \in \Lambda$, the sets

$$A_n(x) = f^{-n}(\mathcal{F}^s_r(f^n(x))) \quad \text{and} \quad B_n(x) = f^n(\mathcal{F}^u_r(f^{-n}(x)))$$

yield nested sequences: $A_n(x) \subset A_{n+1}(x)$ and $B_n(x) \subset B_{n+1}(x)$ for all $n \in \mathbb{N}$. Moreover,

$$\mathcal{F}^s(x) = \bigcup_{n\in\mathbb{N}} A_n(x) \quad \text{and} \quad \mathcal{F}^u(x) = \bigcup_{n\in\mathbb{N}} B_n(x).$$

Proof:

We only prove the statement for the sets $A_n(x)$ (the proof for $B_n(x)$ is identical).

Recall that we are assuming that the norm $||.||$ is adapted. Thus, by the partial hyperbolicity of Λ we have that

$$||Df_{x|_{E^s}}|| < \lambda < 1 \quad \text{for every} \quad x \in \Lambda, \quad \text{(see Definition 3.1)}.$$

Fixed $x \in \Lambda$, there is $r_x > 0$ sufficiently small so that

$$\mathcal{F}^s_{r_x}(f(x)) \supset f(\mathcal{F}^s_{r_x}(x)). \tag{3.0.1}$$

Observe that, since f is C^1, we can choose r_x depending continuously on x. As Λ is compact, we can replace r_x to a uniform constant $r = \min\{r_x \mid x \in \Lambda\}$ in Equation (3.0.1). Thus,

$$\mathcal{F}^s_r(f(x)) \supset f(\mathcal{F}^s_r(x)) \quad \text{for every } x \in \Lambda. \tag{3.0.2}$$

Denoting $y = f^n(x)$, Equation (3.0.2) immediately implies that

$$A_{n+1}(x) = f^{-n-1}(\mathcal{F}^s_r(f(y))) \supset f^{-n}(\mathcal{F}^s_r(y)) = A_n(x).$$

This proves the first part of the lemma. For the last part, we first observe that $A_n(x) \subset \mathcal{F}^s(x)$ for every $n \in \mathbb{N}$. Then we obtain $\bigcup_{n\in\mathbb{N}} A_n(x) \subset \mathcal{F}^s(x)$. To get the inverse inclusion we first write

$$\mathcal{F}^s(x) = \bigcup_{\delta>0} \mathcal{F}^s_\delta(x).$$

Fix $\delta > 0$. The exponential contraction on the strong stable leaves gives that, for n sufficiently large, $f^n(\mathcal{F}^s_\delta(x)) \subset \mathcal{F}^s_r(f^n(x))$. Hence, $\mathcal{F}^s_\delta(x) \subset f^{-n}(\mathcal{F}^s_r(f^n(x))) = A_n(x)$. Since it holds for every $\delta > 0$, we finally obtain

$$\mathcal{F}^s(x) = \bigcup_{\delta>0} \mathcal{F}^s_\delta(x) \subset \bigcup_{n\in\mathbb{N}} A_n(x).$$

■

Remark 3.7 A particular situation is when the leaves of the foliation is a subset of Λ. It occurs, for instance, when $\Lambda = \Lambda_f(U)$ is an attractor. In this case the leaves of $\mathcal{F}^u$ and the unstable manifolds are subsets of Λ, and the constant C_2 in Proposition 3.4 can be chosen uniformly.

To see this Remark, consider a point $x \in U$ and let W be a strong unstable disk or a local unstable manifold that is sufficiently small so that $W \subset U$ and $f(W) \subset U$. Then $f^{n+1}(W) \subset f^n(U)$ for every $n \in \mathbb{N}$. By the continuous dependence of the leaves and Lemma 3.6, the orbit of every accumulation point z of the forward orbit of x is contained in U. Since it holds for every x, we obtain that the orbit of an unstable leaf (or manifold) lies in U. Hence, it is in the maximal invariant subset $\Lambda_f(U)$ of U. The uniformity of C_2 follows from the partial hyperbolicity on Λ, since these leaves are subsets of Λ.

Remark 3.8 If there are saddles $p, q \in \Lambda$ of indices s and $s + c$, respectively, then $W^s(p) = \mathcal{F}^s(p)$ and $W^u(q) = \mathcal{F}^u(q)$. For any sufficiently small $\varepsilon > 0$, we get that $\{p\} = \mathcal{F}^s_\varepsilon(p) \pitchfork W^u_\varepsilon(p)$ and $\{q\} = W^s_\varepsilon(q) \pitchfork \mathcal{F}^u_\varepsilon(q)$. Then, from the continuity of the foliations, it holds that

1. If $y \in \Lambda$ is close enough to p then $\mathcal{F}^s(y) \pitchfork W^u_\varepsilon(p) \neq \emptyset$,

2. If $y \in \Lambda$ is close enough to q then $\mathcal{F}^u(y) \pitchfork W^s_\varepsilon(q) \neq \emptyset$.

Remark 3.9 Suppose that the forward (resp. backward) orbit of x is dense in Λ. Then the forward (resp. backward) orbit of any point $z \in \mathcal{F}^s(x)$ (resp. $\mathcal{F}^u(x)$) is dense in Λ. It is a straightforward consequence of (3) in Proposition 3.4.

When dealing with perturbations of a diffeomorphism, as in the case of the continuations of isolated sets, we need to specify in these notations which diffeomorphism we are referring to. So, let $\Lambda_f(U)$ be an isolated (s,c,u)-partially hyperbolic set and $\mathcal{U}$ be a neighborhood of f such that, for every $g \in \mathcal{U}$, the set $\Lambda_g(U)$ is (s,c,u)-partially hyperbolic (that is, the dimensions of the sub-bundles in the partially hyperbolic splitting on $\Lambda_g(U)$ are the same as on $\Lambda_f(U)$). We denote by $\mathcal{F}^s(g)$ and by $\mathcal{F}^s(x,g)$, respectively, the strong stable foliation of $\Lambda_g(U)$ (with respect to the partial hyperbolicity of g) and the leaf of this foliation that contains x. Similarly, given a hyperbolic periodic point $x \in \Lambda_g(U)$ and $\varepsilon > 0$, we denote by $W^s_\varepsilon(x,g)$ and $W^s(x,g)$ the local and the global stable manifolds of x. The union of all local or all global stable manifolds along the orbit of x is denoted by $W^s_\varepsilon(\mathcal{O}_g(x),g)$ and $W^s(\mathcal{O}_g(x),g)$, respectively.

Similar notations are considered for the unstable foliation and manifold.

Remark 3.10 The leaves of $\mathcal{F}^s(g)$ and $\mathcal{F}^u(g)$ depend continuously on the diffeomorphism g. This means that, fixed $r > 0$ and $\varepsilon > 0$, if g is sufficiently close to f and $x \in \Lambda_f(U)$ is sufficiently close to $y \in \Lambda_g(U)$ then the disk $\mathcal{F}^s_r(y,g)$ is ε close to the disk $\mathcal{F}^s_r(x,f)$ (in the Hausdorff distance).

Definition 3.11 Let $\Lambda_f(U)$ be an isolated set of a diffeomorphism $f \in \mathrm{Diff}^1(M)$. We call a neighborhood $\mathcal{U}$ of f a *compatible* neighborhood if $\mathcal{U}$ is sufficiently small so that, for all $g \in \mathcal{U}$, we have:

- the continuation $\Lambda_g(U)$ of $\Lambda_f(U)$ is well defined,
- the set $\Lambda_g(U)$ is an isolated set,
- if $\Lambda_f(U)$ is an attractor of f, then $\Lambda_g(U)$ is an attractor of g,
- if $\Lambda_f(U)$ is a (s,c,u)-partially hyperbolic set then $\Lambda_g(U)$ is a (s,c,u)-partially hyperbolic set of g,
- if $\Lambda_f(U)$ is a generically (resp. robustly) transitive set of f, then $\Lambda_g(U)$ is a generically (resp. robustly) transitive set of g.