5 Minimality

Here we introduce the notion of minimality for "set foliations". This concept extends the one given in (12) to the case of partially hyperbolic diffeomorphisms. The idea is essentially the same: we requires that the orbit of the leaves accumulates all over the set. However, in the case of proper sets, this accumulation could be done "outside" the set (that is, with a small or none intersection with the set). Here we ask for a more stronger accumulation property that is done "inside" the set, see Definition 5.1. We also observe that we do not consider the whole orbit of the leaves, but just a finite piece of orbit accumulating the set. This last assumption is related with a decomposition of the set into a finite number of smaller indecomposable ones, as we explain in section 7.

5.1 General properties of u and s-minimal sets

Here we give a precise definition of minimality for "set foliations" and state the main properties of these sets. The results in this section do not require the set to be an attractor neither the central bundle to be one dimensional.

For notational simplicity, given a strongly partially hyperbolic set Λ we adopt the following notation.

$$\mathcal{F}^s_\Lambda(x) = \mathcal{F}^s(x) \cap \Lambda \quad \text{and} \quad \mathcal{F}^u_\Lambda(x) = \mathcal{F}^u(x) \cap \Lambda.$$

Definition 5.1 (s and u-minimal foliations) Let Λ be a strongly partially hyperbolic set of a diffeomorphism f. The foliation $\mathcal{F}^s$ is *minimal* if there is $d \in \mathbb{N}$ such that for all $x \in \Lambda$ it holds that

$$\bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(x))} = \Lambda.$$

In this case we say that Λ is an *s-minimal set*.

If Λ is an isolated set with $\Lambda = \Lambda_f(U)$, then we say that Λ is a robustly s-minimal set if $\Lambda_g(U)$ is s-minimal for all g in a neighborhood $\mathcal{U}$ of f.

The definition of u-minimality is analogous considering the strong unstable foliation $\mathcal{F}^u$.

The reason why the constant d appears in this definition is explained in Section 7. Roughly speaking, this constant gives the maximal finite partition of Λ into compact subsets that are cycled permuted by the action of f (see Definition 7.1 for a more precise description). We also anticipate that in the case of robustly u or s-minimal attractors, the constant d can be taken uniformly in a neighborhood of f (see Theorem 7.7).

In what follows, we gather some consequences of u and s-minimality concerning the behaviour of the orbit of a strong stable or unstable disk. The main result in this section is the following.

Theorem 5.2 *Any s-minimal (or u-minimal) set with non-empty interior is the whole manifold.*

For close related results about transitive sets with non-empty interior see, for instance, (2) and (4). These results requires either generic arguments or that the partial hyperbolicity holds in the whole manifold M. Observe that the statement of Theorem 5.2 do not require genericity, a fact that will be important to extract open properties.

Corollary 5.3 *Any robustly s-minimal (or u-minimal) proper set $\Lambda_f(U)$ (i.e., with $U \neq M$) has robustly empty interior.*

In the rest of this section, all the results are stated for s-minimal sets. Similar results (with similar proofs) also hold in the u-minimal case.

Given a set $K \subset M$, we denote by $B_\varepsilon(K)$ the ε-neighborhood of K in M. That is, $B_\varepsilon(K)$ is the set of all points in M whose distance to K is less than ε.

Lemma 5.4 *Let Λ be an s-minimal set of a diffeomorphism f and d be as in the definition of s-minimality. Given $\varepsilon > 0$ and $r > 0$, there is a constant $N = N(\varepsilon, r) \in \mathbb{N}$ such that*

$$\Lambda \subset B_\varepsilon\Big(\bigcup_{i=1}^{d} f^{-k+i}(\mathcal{F}^s_r(x))\Big) \quad \textit{for all } x \in \Lambda \textit{ and } k > N.$$

Proof: From s-minimality and Lemma 3.6, given any $y \in \Lambda$, there is some $N_y \in \mathbb{N}$ such that

$$\Lambda \subset B_\varepsilon\Big(\bigcup_{i=1}^{d} f^{i}(f^{-N_y}(\mathcal{F}^s_r(f^{N_y}(y))))\Big).$$

From the continuity of the foliation $\mathcal{F}^s$, there is a neighborhood $V(y)$ of y such that the previous inclusion holds for all $z \in V(y) \cap \Lambda$. Consider the covering $\{V(y)\}_{y\in\Lambda}$ of Λ. By the compactness of Λ, we may extract a finite subcovering $\{V(y_i)\}_{i=1}^m$ and constants $N_i = N_{y_i}$ such that, if $y \in \Lambda \cap V(y_j)$ for some $j \in \{1, \ldots, m\}$, then

$$\Lambda \subset B_\varepsilon\Big(\bigcup_{i=1}^{d} f^i(f^{-N_j}(\mathcal{F}_r^s(f^{N_j}(y))))\Big).$$

By Lemma 3.6, this inclusion stills holds replacing N_j by any number $N \geq N_j$. Setting $N = \max\{N_1, \ldots, N_m\}$, we have

$$\Lambda \subset B_\varepsilon\Big(\bigcup_{i=1}^{d} f^i(f^{-N}(\mathcal{F}_r^s(f^{N}(y))))\Big),$$

for every $y \in \Lambda$. Letting $x = f^N(y)$, we obtain the desired inclusion. ■

Lemma 5.5 *Let Λ be an s-minimal set of a diffeomorphism f. If Λ contains some strong stable disk then it contains the strong stable leaf of any point in Λ.*

Proof: Let $r > 0$ and $x_0 \in \Lambda$ be such that the strong stable disk $D = \mathcal{F}_r^s(x_0)$ is a contained in Λ. Applying Lemma 5.4 to r and $\varepsilon_k = 1/k$, we get that $\bigcup_{j=1}^{\infty} f^{-j}(D)$ is a $(1/k)$-dense subset of Λ. As this holds for all $k \in \mathbb{N}$ this set is dense in Λ.

Claim 5.6 *Assume that $y \in \Lambda$ is accumulated by the backward orbit of x_0. Then $\bigcup_{i=1}^{d} f^i(\mathcal{F}^s(y))$ is a dense subset of Λ.*

Proof: Fix $\delta > r$. Since $D \subset \Lambda$, this disk is uniformly expanded by backward iterations of f. Then there is an increasing sequence $\{n_i\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_{i\to\infty} f^{-n_i}(x_0) = y$ and, for every $i \in \mathbb{N}$, the disk $f^{-n_i}(D)$ has inner radius bigger than δ. By the continuity of the foliation, we obtain that $\mathcal{F}_\delta^s(y) \subset \Lambda$. As this holds for every $\delta > r$, the whole leaf $\mathcal{F}^s(y)$ is contained in Λ. Now s-minimality implies that $\bigcup_{i=1}^{d} f^i(\mathcal{F}^s(y))$ is a dense subset of Λ. ■

As a consequence of this claim, every $z \in \Lambda$ is accumulated by entire strong stable leaves contained in Λ. Again by the continuity of the strong stable foliation and the closeness of Λ we conclude that $\mathcal{F}^s(z) \subset \Lambda$. ■

We are now ready to finish the proof of Theorem 5.2

ProofProof of Theorem 5.2: The interior of Λ, $\mathrm{int}(\Lambda)$, is an invariant subset of Λ. Moreover, if Λ has non-empty interior then it contains some stable disk and by Lemma 5.5 it contains all of its strong stable leaves.

If the frontier $\partial\Lambda$ of Λ is empty we are done. Otherwise, given $z \in \partial\Lambda$ and any disk $D = \mathcal{F}^s_r(z)$, Lemma 5.4 implies that there is $N \in \mathbb{N}$ such that $f^{-N}(D)$ intersects $\mathrm{int}(\Lambda)$. The f-invariance of the interior of Λ implies that $D \cap \mathrm{int}(\Lambda) \neq \emptyset$. Now, choose some point x in this intersection and an open neighborhood B of x with $B \subset \mathrm{int}(\Lambda)$. For each point $y \in B$ we consider its entire strong stable leave $\mathcal{F}^s(y)$, that is contained in Λ (recall Lemma 5.5). By the continuity of the strong stable foliation, the set $V = \bigcup_{y\in B} \mathcal{F}^s(y) \subset \Lambda$ is a neighborhood of $\mathcal{F}^s(x) = \mathcal{F}^s(z)$. Thus V is a neighborhood of z that is contained in Λ, contradicting the fact that $z \in \partial\Lambda$. Hence $\partial\Lambda = \emptyset$, a contradiction. ■

5.2 Pertubations of isolated u and s-minimal sets

In this subsection we study the effect of minimality for the continuations of isolated sets. We see how minimal sets are related with “their homoclinic classes”. Here we only deal with partially hyperbolic sets with one dimensional central bundle. As in Subsection 5.1, the results are stated only for s-minimal sets, although dual versions hold in the u-minimal case.

Theorem 5.7 *Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be an isolated $(s,1,u)$-partially hyperbolic set that is s-minimal. Let $\mathcal{U}$ be a compatible neighborhood of f. Then for every hyperbolic periodic point $p \in \Lambda_f(U)$ there is an open set $\mathcal{W}_p \subset \mathcal{U}$, with $f \in \overline{\mathcal{W}_p}$, such that for all $g \in \mathcal{W}_p$ one has*

$$H(p_g, g) \subset \overline{\mathcal{O}^-_g(D)},$$

where D is any strong stable disk centered at some point $x \in \Lambda_g(U)$. In addition, if the index of p is s, then $\mathcal{W}_p$ can be taken being a neighborhood of f.

Proof: By Remark 2.2 the isolated sets vary upper semicontinuously, so we can assume that every diffeomorphism in $\mathcal{R}$ is a continuity point of the map $g \mapsto \Lambda_g(U)$.

Fix a hyperbolic periodic point $p \in \Lambda_f(U)$. Assume first that p has index s and consider, for a small ε, the local unstable manifold $W^u_\varepsilon(\mathcal{O}_f(p))$ of the orbit of p. By s-minimality, for each $x \in \Lambda_f(U)$ the leaf $\mathcal{F}^s(x)$ intersects transversely $W^u_\varepsilon(\mathcal{O}_f(p))$. Hence, by the continuity of the strong stable foliation, for each $x \in \Lambda_f(U)$ there are neighborhoods U_x of x and $\mathcal{V}_x$ of f such that

$\mathcal{F}^s(y,g)$ intersects transversely $W^u_\varepsilon(\mathcal{O}_g(p_g),g)$ for every $y \in U_x \cap \Lambda_g(U)$ and every $g \in \mathcal{V}_x$.

Using the open sets U_x we get a finite covering $U_{x_1},\dots,U_{x_m}$ of $\Lambda_f(U)$. Write $B = \bigcup_{i=1}^m U_{x_i} \supset \Lambda_f(U)$ and $\mathcal{V} = \bigcap_{i=1}^m \mathcal{V}_{x_i}$. Since f is a continuity point of the map $g \mapsto \Lambda_g(U)$, after shrinking $\mathcal{V}$ if necessary, we have that $\Lambda_g(U) \subset B$ for all $g \in \mathcal{V}$. By construction, for every point $y \in \Lambda_g(U)$ and every $g \in \mathcal{V}$ we have that $\mathcal{F}^s(y,g)$ intersects transversely $W^u_\varepsilon(\mathcal{O}(p_g),g)$.

Applying the λ-lemma, this implies that

$$W^s(\mathcal{O}_g(p_g),g) = \mathcal{F}(\mathcal{O}_g(p_g),g) \subset \overline{\mathcal{O}_g(\mathcal{F}^s(y,g))},$$

and thus

$$H(p_g,g) \subset \overline{\mathcal{O}_g(\mathcal{F}^s(p_g,g))} \subset \overline{\mathcal{O}_g(\mathcal{F}^s(y,g))}. \tag{5.2.1}$$

To conclude the proof of the theorem (for the case of index$(p) = s$) it is enough to observe that if D is any strong stable disk centered at some point $x \in \Lambda_g(U)$ and y is an accumulation point of the pre-orbit of x, then

$$\overline{\mathcal{O}_g(\mathcal{F}^s(y,g))} \subset \overline{\mathcal{O}_g^-(D)}.$$

By Equation (5.2.1), setting $\mathcal{W}_p = \mathcal{V}$ we obtain the desired neighborhood of f.

When index of p is $s+1$, consider another periodic point $q \in \Lambda_f(U)$ with index s (the existence of such point is assured by (2) of Proposition 4.4). Let $\mathcal{V}_{p,q}$ be the open set associated to p and q given by Proposition 4.9. By the first part of the proof there is a neighborhood $\mathcal{W}_q$ of f such that for every $g \in \mathcal{W}_q$ it holds

$$\overline{\mathcal{O}_g(\mathcal{F}^s(q_g,g))} \subset \overline{\mathcal{O}_g^-(D)}.$$

By item (1) of Proposition 4.9, if $g \in \mathcal{V}_{p,q}$ we also have that

$$H(p_g,g) \subset \overline{W^s(\mathcal{O}_g(p_g),g)} \subset \overline{\mathcal{O}_g(\mathcal{F}^s(q_g,g))} \subset \overline{\mathcal{O}_g^-(D)}.$$

We conclude the proof by setting $\mathcal{W}_p = \mathcal{W}_q \cap \mathcal{V}_{p,q}$. Recall that $f \subset \overline{\mathcal{V}_{p,q}}$ and $\mathcal{W}_q$ is a neighborhood of f, so we have that $f \in \overline{\mathcal{W}_p}$. ∎

In Proposition 2.4 of (11) it is proved that the continuations of a robustly transitive set $\Lambda_f(U)$ coincide with the relative homoclinic class of a periodic point in a locally residual neighborhood of f. That is, a hyperbolic periodic point p of $\Lambda_f(U)$, a neighborhood $\mathcal{U}$ of f, and a residual subset $\mathcal{T}$ of $\mathcal{U}$ such that

$$H_U(p_g,g) = \Lambda_g(U) \quad \text{for all } g \in \mathcal{T}.$$

Remark 5.8 This fact together with Remark 4.6 implies that, C^1-generically, a transitive attractor coincides with a homoclinic class.

Under the additional assumption of s-minimality, we see in the next theorem that we can characterize robustly transitive sets in terms of its transverse homoclinic points in a robust way: these sets coincide robustly with the set $\{x \in H(p,f) \,|\, \mathcal{O}_f(x) \subset U\}$. Compare this set with the definition of $H_U(p,f)$. This is equivalent to say that $\Lambda_f(U)$ is, robustly, a subset of $H(p,f)$. In the case of attractors, Theorem 5.9 implies that Remark 5.8 holds robustly, see Corollary 6.3.

Theorem 5.9 *Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be a robustly transitive set that is $(s,1,u)$-partially hyperbolic and s-minimal. Then, given any hyperbolic periodic point $p \in \Lambda_f(U)$, there is an open set $\mathcal{W}_p \subset \mathrm{Diff}^1(M)$, with $f \in \overline{\mathcal{W}_p}$, such that $\Lambda_g(U) \subset H(p_g,g)$ for all $g \in \mathcal{W}_p$.*

Proof: Fix a hyperbolic periodic point $p \in \Lambda_f(U)$. By Theorem 5.7, we can assume that there is an open subset $\mathcal{W}_p$ of $\mathrm{Diff}^1(M)$, with $f \in \overline{\mathcal{W}_p}$, such that, for every $g \in \mathcal{W}_p$ and every strong stable disk D centered at some point of $\Lambda_g(U)$, it holds that $H(p_g,g) \subset \overline{\mathcal{O}_g^-(D)}$. From item (2) and (3) in Proposition 4.9, we can assume that index of p is s and that

$$\Lambda_g(U) \subset \overline{W^s(\mathcal{O}_g(p_g),g)} = \overline{\mathcal{O}_g(\mathcal{F}^s(p_g,g))} \quad \text{for all} \quad g \in \mathcal{W}_p. \tag{5.2.2}$$

Let $x \in \Lambda_g(U)$ be a point of forward transitive orbit. Given $y \in \Lambda_g(U)$ and $\varepsilon > 0$, consider $n_1, n_2 \in \mathbb{N}$ satisfying

1. $g^{n_1}(x)$ is sufficiently close to p_g so that its strong stable leaf cut the local unstable manifold of p_g at a point z, and
2. $g^{n_2}(x)$ is $\varepsilon/3$-close to y, and n_2 is sufficiently large so that $g^{n_2}(x)$ is $\varepsilon/3$-close to $g^{n_2}(z)$.

From Equation (5.2.2) and Proposition 4.14 the orbit of $\mathcal{F}^s(p_g)$ accumulates at $g^{n_2}(x)$ and then meet transversely $W^u(p_g,g)$ in a point w that is $\varepsilon/3$-close to $g^{n_2}(z)$. Then, w is a transverse homoclinic point that is ε-close to y. From the arbitrary choice of ε, we obtain that $y \in H(p_g,g)$. Since it holds for every $y \in \Lambda_g(U)$, we conclude that $\Lambda_g(U) \subset H(p_g,g)$. ■

The next two lemmas concerns the behaviour of the strong leaves when Λ has a minimal foliation. The first one shows that the invariant manifolds that "contain" the central direction intersect transversely any strong leaf of the minimal foliation and, in particular, contain a dense subset of Λ. The second one shows that the accumulation of one leaf at another one can be restricted to the set Λ.

Lemma 5.10 *Let $f \in \mathcal{R}$ and $\Lambda = H(p,f)$ be an isolated s-minimal (s,c,u)-partially hyperbolic homoclinic class of a hyperbolic periodic point p of index s. Then, the unstable manifold of p meets transversely any strong stable disk centered at a point in Λ.*

Proof: Fix $x \in \Lambda_f(U)$, $r > 0$, $\varepsilon > 0$, and $\delta > 0$. By Lemma 5.4, if $\varepsilon > 0$ is sufficiently small, there is $N \in \mathbb{N}$ such that $f^{-N}(\mathcal{F}^s_r(x))$ is δ-close to p. By Proposition 4.14, extending $f^{-N}(\mathcal{F}^s_r(x))$ to a δ-neighborhood $\mathcal{S}$ inside the leaf $\mathcal{F}^s(f^{-N}(x))$, the disk $\mathcal{S}$ intersect transversely $W^u_\varepsilon(\mathcal{O}_f(p))$.

Let $t > r$ be such that $f^N(\mathcal{S}) = \mathcal{F}^s_t(x)$. Thus,

$$\mathcal{F}^s_t(x) \pitchfork f^N(W^u_\varepsilon(\mathcal{O}_f(p)) \neq \emptyset. \tag{5.2.3}$$

We can choose N big enough so that, by the exponential contraction on the disk $\mathcal{S}$ (see item (3) of Proposition 3.4), the estimation $t < 2r$ holds. Since we can choose r arbitrarily small, Equation (5.2.3) and this estimation imply that $W^u(\mathcal{O}_f(p))$ meets transversely any strong stable disk centered at x. ■

Lemma 5.11 *Let $f \in \mathcal{R}$ and $\Lambda = H(p,f)$ be an isolated s-minimal (s,c,u)-partially hyperbolic set of some hyperbolic periodic point p of index s. Then, for every $x, y \in \Lambda$ satisfying $\mathcal{F}^s(x) \subset \overline{\mathcal{F}^s(y)}$ it holds that $\mathcal{F}^s_\Lambda(x) \subset \overline{\mathcal{F}^s_\Lambda(y)}$.*

Proof: Let $z \in \mathcal{F}^s_\Lambda(x)$, $r > 0$ and consider the disk $\mathcal{F}^s_r(z)$. By Lemma 5.10, $W^u(p)$ meets transversely $\mathcal{F}^s_r(z)$, say at the point w. Since $\mathcal{F}^s(x) \subset \overline{\mathcal{F}^s(y)}$, we also have an intersection $\hat{w}$ of $\mathcal{F}^s(y)$ and $W^u(p)$ that we can choose arbitrarily close to w. From s-minimality, the orbit of $\mathcal{F}^s(p)$ accumulates at $\mathcal{F}^s(y)$ and thus intersect transversely $W^u(p)$ in a sequence of points that accumulate at $\hat{w}$. This sequence of points consist of transverse homoclinic points of p, so $\hat{w} \in \Lambda$. As r can be chosen arbitrarily small and $\hat{w}$ can be chosen arbitrarily close to w, we conclude that $x \in \overline{\mathcal{F}^s_\Lambda(y)}$. Since it holds for every $x \in \mathcal{F}^s(x)$ we finally obtain that $\mathcal{F}^s_\Lambda(x) \subset \overline{\mathcal{F}^s_\Lambda(y)}$. ■

5.3 A criterium for minimality

Next we stablish a criterion to verify u or s-minimality on homoclinic classes.

Let us denote by $\mathrm{Per}_\sigma(f_{|\Lambda})$ the set of periodic points of Λ with index σ.

Theorem 5.12 (generic criterion for minimality) *Let $f \in \mathcal{R}$ and $\Lambda = H(p,f)$ be an $(s,1,u)$-partially hyperbolic isolated homoclinic class. Then,*

1. *If* $\overline{\mathcal{O}_f(\mathcal{F}^s(x))} \cap \mathrm{Per}_s(f_{|\Lambda}) \neq \emptyset$ *for every* $x \in \Lambda$*, then* Λ *is* s*-minimal.*
2. *If* $\overline{\mathcal{O}_f(\mathcal{F}^u(x))} \cap \mathrm{Per}_{s+1}(f_{|\Lambda}) \neq \emptyset$ *for every* $x \in \Lambda$*, then* Λ *is* u*-minimal.*

To prove this criterion we need some auxiliary lemmas. As usual, we only treat the s-minimal case.

Lemma 5.13 *Let* $f \in \mathcal{R}$ *and* $\Lambda = H(p,f)$ *be an isolated* (s,c,u)*-partially hyperbolic set of some hyperbolic periodic point* p *of index* s *and period* d*. If there is* $x \in \Lambda$ *such that* $p \in \overline{\mathcal{F}^s(x)}$*, then*

$$\Lambda = \bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(x))}.$$

Proof:

Note first that the inclusion $\bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(x))} \subset \Lambda$ is immediate (recall the notation $\mathcal{F}^s_\Lambda(x) = \mathcal{F}^s(x) \cap \Lambda$).

To prove that $\Lambda \subset \bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(x))}$ observe that, as $\Lambda = H(p,f)$, the period p is d, and $\mathrm{index}(p) = s$, we have that

$$\Lambda = \bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(p))}. \tag{5.3.1}$$

On the other hand, as $p \in \overline{\mathcal{F}^s(x)}$, Proposition 4.16 gives that $\mathcal{F}^s(p) \subset \overline{\mathcal{F}^s(x)}$.

This last inclusion leads to $\mathcal{F}^s_\Lambda(p) \subset \overline{\mathcal{F}^s_\Lambda(x)}$. Indeed, given $z \in \mathcal{F}^s_\Lambda(p)$, consider a transverse homoclinic point $\tilde{z}$ of p close to z. Since $\mathcal{F}^s(p) \subset \overline{\mathcal{F}^s(x)}$, the leaf $\mathcal{F}^s(x)$ accumulates at $\tilde{z}$ and intersect $W^u(\mathcal{O}_f(p))$ at a point w that can be chosen arbitrarily close to $\tilde{z}$. By Equation (5.3.1), $\mathcal{F}^s(p)$ accumulates at x and, consequently, it also accumulates at w. Then, $\mathcal{F}^s(p)$ meet $W^u(\mathcal{O}_f(p))$ in a sequence of points converging to w. Hence, $w \in H(p,f)$ and, as w can be obtained arbitrarily close to z, we conclude that $z \in \overline{\mathcal{F}^s_\Lambda(x)}$. From the arbitrary choice of $z \in \mathcal{F}^s_\Lambda(p)$, we obtain that $\mathcal{F}^s_\Lambda(p) \subset \overline{\mathcal{F}^s_\Lambda(x)}$.

This last inclusion and Equation (5.3.1) leads to

$$\Lambda \subset \bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(x))},$$

finishing the proof. ■

Lemma 5.14 *Under the notation of Lemma 5.13, assume that*

$$p \in \overline{\mathcal{O}(\mathcal{F}^s(x))} \quad \text{for all } x \in \Lambda.$$

Then Λ *is* s*-minimal.*

Proof: Fix $x \in \Lambda$. Since $p \in \overline{\mathcal{O}(\mathcal{F}^s(x))}$, given $\varepsilon > 0$ there is $j_1 \in \mathbb{Z}$ such that

$$p \in B_\varepsilon(\mathcal{F}^s(f^{j_1}(x))).$$

By Proposition 4.14, the leaf $\mathcal{F}^s(f^{j_1}(x))$ intersects transversely the local unstable manifold of p provided ε is small enough. Hence, by the λ-lemma, there is $j_2(x) \in \mathbb{N}$ such that, for every $j \geq j_2(x)$, it holds that

$$\Lambda \subset B_{\frac{\varepsilon}{2}}\Big(\bigcup_{i=1}^{d} \mathcal{F}^s(f^{-j+i}(x))\Big), \quad \text{where } d \text{ is the period of } p.$$

By the continuity of the foliation $\mathcal{F}^s$ there is a neighborhood U_x of x satisfying

$$\Lambda \subset B_\varepsilon\Big(\bigcup_{i=1}^{d} \mathcal{F}^s(f^{-j+i}(y))\Big), \quad \text{for all } y \in U_x \cap \Lambda. \tag{5.3.2}$$

In this way, for each $x \in \Lambda$ we get a number $j_2(x)$ and a neighborhood U_x of x satisfying Equation (5.3.2). Using these open sets we can get a finite covering $\bigcup_{i=1}^{m} U_{x_i}$ of Λ.

Set $J = \max_{i=1}^{m}\{j_2(x_i)\}$. Then, by construction,

$$\Lambda \subset B_\varepsilon\Big(\bigcup_{i=1}^{d} \mathcal{F}^s(f^{-J+i}(y))\Big), \quad \text{for all } y \in \Lambda.$$

Fix $x \in \Lambda$ and let $y \in \Lambda$ be such that $x = f^{-J}(y)$. As ε can be taken arbitrarily small, we conclude that $\Lambda \subset \bigcup_{i=1}^{d} \overline{\mathcal{F}^s(f^i(x))}$. Then, there is $j_3(x) \in \{1, ..., d\}$ such that $p \in \overline{\mathcal{F}^s(f^{j_3(x)}(x))}$, which is equivalent to $f^{-j_3(x)}(p) \in \overline{\mathcal{F}^s(x)}$.

Applying Lemma 5.13 to this last inclusion, and observing that $\Lambda = H(p, f) = H(f^{-j_3(x)}(p), f)$, we obtain that

$$\bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(x))} = \Lambda.$$

As it holds for all $x \in \Lambda$, the set Λ is s-minimal. ■

Remark 5.15 The dual statement for the unstable foliation holds for homoclinic classes of periodic points of index $s + 1$.

Remark 5.16 *In the case of homoclinic classes that are attractors, Lemmas 5.10, 5.11, 5.13, and 5.14 hold not only in $\mathcal{R}$, but for every $f \in \mathrm{Diff}^1(M)$. This is so because the only part of the proof that we need some genericity is to use Proposition 4.14. By Remark 4.12, Proposition 4.14 holds more generally in the case of attractors.*

ProofProof of Theorem 5.12: Fix two hyperbolic periodic points p_1 and p_2 of Λ with indices s and $s+1$, respectively. We first prove item (1). By hypothesis, for every $x \in \Lambda$ there is a point $p_x \in \overline{\mathcal{O}(\mathcal{F}^s(x))} \cap \mathrm{Per}_s(f_{|\Lambda})$. Then, from Proposition 4.16 and the invariance of the set $\overline{\mathcal{O}(\mathcal{F}^s(x))}$, we have that

$$\overline{\mathcal{O}_f(\mathcal{F}^s(p_x))} \subset \overline{\mathcal{O}_f(\mathcal{F}^s(x))}. \tag{5.3.3}$$

Since $p_x \in \mathrm{Per}_s(f_{|\Lambda})$, we have that $\mathcal{F}^s(p_x) = W^s(p_x)$, and thus

$$H(p_x, f) \subset \overline{\mathcal{O}_f(\mathcal{F}^s(p_x))}. \tag{5.3.4}$$

By item (5) of Theorem 4.3, every non-disjoint homoclinic class coincide, so $\Lambda = H(p_x, f)$. Now, putting together this fact, equation (5.3.3), and equation (5.3.4) we obtain

$$p_1 \in \Lambda \subset \overline{\mathcal{O}(\mathcal{F}^s(x))}$$

Since this holds for every $x \in \Lambda$, Lemma 5.14 implies the s-minimality of Λ.

For item (2) we follow the same argument, using p_2 and the dual version of Lemma 5.14 for the unstable case. ■

Corollary 5.17 *C^1-generically, a robustly (resp. generically) transitive attractor $\Lambda_f(U)$ that is $(s,1,u)$-partially hyperbolic and s-minimal is robustly (resp. generically) s-minimal.*

Proof: Let us first consider the generic case. Let $p \in \Lambda_f(U)$ be a periodic point of index s (given by item (2) of Proposition 4.4) and $\mathcal{W}_p$ be the neighborhood of f given in Theorem 5.7. Then, for every $g \in \mathcal{W}_p$ and every disk $D = \mathcal{F}^s_r(x, g)$ with $x \in \Lambda_g(U)$ it holds that $H(p_g, g) \subset \overline{\mathcal{O}^-_g(D)}$.

By Remark 5.8, a transitive attractor is, generically, a homoclinic class. Hence there is a residual subset $\mathcal{Z}$ of $\mathcal{W}_p$ such that, for every $g \in \mathcal{Z}$, it holds that

$$\Lambda_g(U) \subset \overline{\mathcal{O}^-_g(D)}$$

for every disk $D = \mathcal{F}^s_r(x, g)$ with $x \in \Lambda_f(U)$. In particular it holds that

$$\Lambda_g(U) \subset \overline{\mathcal{O}(\mathcal{F}^s(x, g))}$$

for every $x \in \Lambda_g(U)$. By Theorem 5.12, $\Lambda_g(U)$ is s-minimal, ending the proof for the generic case.

The case of robustly transitive sets is analogous. We just replace the residual set $\mathcal{Z}$ with the open and dense set given by Corollary 6.3, where the continuations of the attractors are, robustly, homoclinic classes. ■

Theorem 5.18 *Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be a $(s,1,u)$-partially hyperbolic attractor that is both u and s-minimal. Then $\Lambda_f(U)$ is robustly u and s-minimal.*

Proof:

Since $f \in \mathcal{R}$, we can take a pair of hyperbolic periodic points $p, q \in \Lambda_f(U)$ satisfying item (1) of Proposition 4.9 in an open neighborhood of f.

Claim 5.19 *There are two neighborhoods $\mathcal{V}_u$ and $\mathcal{V}_s$ of f such that*

$$\Lambda_g(U) = \overline{W^u(\mathcal{O}_g(p_g), g)} \quad \text{for all} \quad g \in \mathcal{V}_u \text{ , and} \tag{5.3.5}$$

$$\Lambda_g(U) = \overline{W^s(\mathcal{O}_g(q_g), g) \cap \Lambda_g(U)} \quad \text{for all} \quad g \in \mathcal{V}_s. \tag{5.3.6}$$

ProofProof of the claim:

We first prove Equation (5.3.5). Note that, as $\Lambda_g(U)$ is an attractor, $W^u(\mathcal{O}_g(p_g), g) \subset \Lambda_g(U)$. then, it suffices to prove that $\Lambda_g(U) \subset \overline{W^u(\mathcal{O}_g(p_g), g)}$.

Consider the neighborhood $\mathcal{W}_p$ of f given by Theorem 5.7, relative to the periodic point p. Fix $g \in \mathcal{W}_p$ and an open set A intersecting $\Lambda_g(U)$. We want to prove that $W^u(\mathcal{O}_g(p_g), g) \cap A \neq \emptyset$. This will imply that $\Lambda_g(U) \subset \overline{W^u(\mathcal{O}_g(p_g), g)}$.

Take $x \in \Lambda_g(U) \cap A$ and consider a strong stable disk $D \subset A$ centered at x. By Theorem 5.7 we have that $H(p_g, g) \subset \overline{\mathcal{O}_g^-(D)}$. In particular the pre-orbit of D accumulates on p_g and, by Proposition 4.14, it must intersect $W^u(p_g, g)$. This implies that the forward orbit of $W^u(p_g, g)$ intersect D and, consequently, it intersects A. From the arbitrary choice of A, we conclude that $\Lambda_g(U) \subset \overline{W^u(\mathcal{O}_g(p_g), g)}$.

The proof of Equation (5.3.6) is very similar to the one of Equation (5.3.5), considering the dual version of Theorem 5.7 for u-minimal sets and the periodic point q of index $s+1$. We only observe that, as $\Lambda_g(U)$ is an attractor, any strong unstable disk is containded in $\Lambda_g(U)$. Then the intersection of the forward orbit of an unstable disk D with $W^s(q_g, g)$ lies in $\Lambda_g(U)$.
■

By this claim, for every $g \in \mathcal{V}_u \cap \mathcal{V}_s$ it holds that

$$\Lambda_g(U) = \overline{W^s(\mathcal{O}_g(q_g), g)} \cap \overline{W^u(\mathcal{O}_g(p_g), g)}.$$

By item (1) of Proposition 4.9, we obtain

$$\Lambda_g(U) = \overline{W^s(\mathcal{O}_g(p_g), g)} \cap \overline{W^u(\mathcal{O}_g(p_g), g)}. \tag{5.3.7}$$

To prove the robust u and s-minimality it is enough to prove that $\Lambda_f(U)$ is robustly transitive, see Corollary 5.17.

Let $g \in \mathcal{V}_u \cap \mathcal{V}_s$. Given two relative open sets A and B of $\Lambda_g(U)$, we need to find $n \in \mathbb{Z}$ such that $g^n(A) \cap B \neq \emptyset$.

Consider $x \in A \cap \Lambda_g(U)$, a small strong stable disk $D \subset A$ centered at x, and some local unstable manifold $W = W^u_\varepsilon(a)$ contained in B, of a hyperbolic periodic point $a \in B$. By Theorem 5.7, the backward orbit of D accumulates at $W^s(\mathcal{O}_g(p_g), g)$, and by Equation (5.3.7), it must accumulates all over $\Lambda_g(U)$. In particular, it accumulates on W, and by Proposition 4.14 it must intersect W at some point z. Since $W \subset \Lambda_g(U)$, the point z lies in B. Hence, there is $n \in \mathbb{N}$ such that $g^{-n}(D) \cap W \neq \emptyset$, and in particular $g^{-n}(A) \cap B \neq \emptyset$. Since it holds for every $g \in \mathcal{V}_s \cap \mathcal{V}_u$, we conclude that f is robustly transitive, finishing the proof. ■

5.4
s-Minimal attractors

In this section we study s-minimal attractors. The main result of this section is the following:

Theorem 5.20 *Let $f \in \mathcal{R}$, $\Lambda_f(U)$ be a $(s, 1, u)$-partially hyperbolic s-minimal proper attractor, and $\mathcal{U}$ be a compatible neighborhood of f. Then there are an open and dense subset $\mathcal{V} \subset \mathcal{U}$ and a residual subset $\mathcal{W}$ of $\mathcal{U}$ such that:*

1. *$\Lambda_g(U)$ has empty interior for all $g \in \mathcal{V}$.*
2. *$\Lambda_g(U)$ has zero Lebesgue measure for all $g \in \mathcal{W}$.*

To prove Theorem 5.20 we need some intermediate lemmas that also hold for $c \geq 1$.

Lemma 5.21 *Let $\Lambda = \Lambda_f(U)$ be an (s, c, u)-partially hyperbolic attractor that is s-minimal, contains some strong stable disk, and has a point $p \in \mathrm{Per_s}(\mathrm{f}_{|\Lambda})$. Then Λ is the whole manifold.*

Proof: By Theorem 5.2, it suffices to prove that Λ has non-empty interior. Consider the periodic point $p \in \mathrm{Per_s}(\mathrm{f}_{|\Lambda})$. Then, for a small $\varepsilon > 0$, its local unstable manifold $W^u_\varepsilon(p)$ is a $(u+c)$-dimensional embedded manifold contained in the attractor. By Lemma 5.5, the strong stable leaf of any point in Λ is contained in Λ. Thus the saturation of $W^u_\varepsilon(p)$ by its strong stable leaves contains an open subset of Λ and hence it has non-empty interior. ■

In what follows, we denote by $\mathrm{Diff}^{1+}(M)$ the subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms whose derivative is α-holder for some $\alpha > 0$. We use the following result that is a simplified version of Corollary B of (4) to the case of partially hyperbolic attractors.

Proposition 5.22 ((4)) *Let $f \in \mathrm{Diff}^{1+}(M)$ and Λ an (s,c,u)-partially hyperbolic set with $\mathrm{Leb}(\Lambda) > 0$. Then Λ contain some strong stable disk and some strong unstable disk.*

Lemma 5.23 *Let $f \in \mathrm{Diff}^{1+}(M)$ and $\Lambda = \Lambda_f(U)$ be an (s,c,u)-partially hyperbolic attractor that is s-minimal. If $\mathrm{Per_s(f_{|\Lambda})} \neq \emptyset$ and $\mathrm{Leb}(\Lambda) > 0$, then Λ is the whole manifold.*

Proof: By Proposition 5.22 there is a strong stable disk D contained in Λ. Now Lemma 5.21 implies the statement. ■

We are now ready to prove Theorem 5.20.

ProofProof of theorem 5.20.: By item (1) of Proposition 4.4, the set $\Lambda_f(U)$ is generically transitive. Let $\mathcal{J}_0$ be the residual subset of $\mathcal{U}$ of diffeomorphisms g such that $\Lambda_g(U)$ is s-minimal given by Corollary 5.17.

Claim 5.24 *For every $g \in \mathcal{J}_0$, $\varepsilon > 0$, and every hyperbolic periodic point $a \in \Lambda_g(U) \cap \mathrm{Per}_{s+1}(g)$ it holds that*

$$\mathrm{int}(W^s_\varepsilon(a) \cap \Lambda_g(U)) = \emptyset.$$

Here the interior refers to the topology of $W^s_\varepsilon(a)$.

Proof: The proof is by contradiction. Assume that there are $\varepsilon > 0$ and $a \in \Lambda_g(U) \cap \mathrm{Per}_{s+1}(g)$ such that $\mathrm{int}(W^s_\varepsilon(a,g) \cap \Lambda_g(U))$ contains an open ball B of $W^s_\varepsilon(a,g)$. By saturating B with strong unstable leaves (which are subsets of the attractor $\Lambda_g(U)$) we get an open set (in the ambient manifold M) contained in $\Lambda_g(U)$. Thus $\Lambda_g(U)$ has non-empty interior and, by Theorem 5.2 it is the whole manifold, contradicting the fact in Remark 2.3 that $\Lambda_g(U)$ is a proper attractor. ■

Consider a diffeomorphism f as in the statement of the theorem and a pair of hyperbolic periodic points $p, q \in \Lambda_f(U)$ with indices s and $s+1$, respectively (recall item (2) of Proposition 4.4 and Remark 3.5). Let $\mathcal{W}_p$ and $\mathcal{V}_{p,q}$ be the open sets given by Proposition 5.7 and 4.9, respectively. By shrinking $\mathcal{W}_p$ is necessary, we can assume that $\mathcal{W}_p \subset \mathcal{V}_{p,q}$, so q_g is well defined for every $g \in \mathcal{W}_p$.

Claim 5.25 *The map ϕ given by $g \mapsto W^s_\varepsilon(q_g, g) \cap \Lambda_g(U)$, defined on $\mathcal{W}_p$, is upper semicontinuous.*

Proof: For every point $z \in W^s_\varepsilon(q_g, g) \cap \Lambda_g(U)$, we have that $z = W^s_\varepsilon(q_g, g) \pitchfork \mathcal{F}^u_\varepsilon(z)$ and $\mathcal{F}^u_\varepsilon(z) \subset \Lambda_g(U)$. This fact together with the continuity of $W^s(p_g, g)$, the continuity of $\mathcal{F}^u_\varepsilon(z)$, and the upper semicontinuity of $\Lambda_g(U)$ in $g \in \mathcal{W}_p$ implies the upper semicontinuity of ϕ. ■

As a consequence this claim, there is a residual subset $\mathcal{J}_1 \subset \mathcal{W}_p$ consisting of continuity points of the map ϕ.

By Claim 5.24 and the definition of $\mathcal{J}_1$ we conclude that, for every $h \in \mathcal{J}_0 \cap \mathcal{J}_1$ (that is a subset of $\mathcal{W}_p$), there is a neighborhood $\mathcal{U}_h$ of h such that

$$W^s_\varepsilon(q_g, g) \not\subset \Lambda_g(U) \quad \text{for all} \quad g \in \mathcal{U}_h. \tag{5.4.1}$$

The set $\mathcal{W} = \bigcup_{h \in \mathcal{J}_0 \cap \mathcal{J}_1} \mathcal{U}_h$ is an open and dense subset of $\mathcal{W}_p$.

Claim 5.26 *For every $g \in \mathcal{W}$, the attractor $\Lambda_g(U)$ do not contain any strong stable disk, and consequently has empty interior.*

Proof: Suppose that there is $g \in \mathcal{W}$ for which $\Lambda_g(U)$ has a strong stable disk $D \subset \Lambda_g(U)$. By the invariance and closeness of $\Lambda_g(U)$, any accumulation point of the backward orbit of D lies inside $\Lambda_g(U)$. By Theorem 5.7, the closure of the negative orbit of D contains $H(p_g, g)$, so we conclude that $\overline{\mathcal{F}^s(p_g, g)} \subset \Lambda_g(U)$. Now, item (1) of Proposition 4.9 implies that $W^s(q_g, g) \subset \Lambda_g(U)$, contradicting Equation (5.4.1).

If $\Lambda_g(U)$ has non-empty interior, then it has some strong stable disk inside its interior. Hence the first part of the proof implies that every $g \in \mathcal{W}$ has empty interior. ■

To obtain item (1) of Theorem 5.20, we apply Claim 5.26 to every diffeomorphism in $\mathcal{R} \cap \mathcal{U}$. The union of all open sets obtained in this way is the announced open and dense subset $\mathcal{V}$ of $\mathcal{U}$.

To prove the second part of the theorem, observe that, if g is a C^2 diffeomorphism in $\mathcal{V}$ such that $\mathrm{Leb}(\Lambda_g(U)) > 0$, then it contains a strong stable disk (see Lemma 5.22). This contradicts Claim 5.26, since we have taken $g \in \mathcal{V}$. Then, for every C^2 diffeomorphisms g in $\mathcal{V}$, the attractor $\Lambda_g(U)$ has zero Lebesgue measure. Since the subset of C^2 diffeomorphisms is dense in $\mathcal{V}$, Corollary 4.20 implies that there is a residual subset of $\mathcal{V}$ where the respective attractors have zero Lebesgue measure. ■