7
Spectral decomposition

In the definition of an s-minimal set Λ we have fixed a natural number d for which the orbit segment of length d of any strong stable leaf is dense in Λ. In this section we analyse the relationship between the constant d and the topological structure of the attractor. We also prove that the constant d can be chosen to be uniform in a neighborhood of a generically or robustly s-minimal attractor. The results in this section also holds for u-minimal sets with analogous proofs. As usual, we only state and prove them for the s-minimal case.

Consider a transitive compact and invariant proper set Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$. The set Λ may consist of the union of a finite number k of pairwise disjoint compact subsets $\{\Lambda_i\}_{i=1}^k$ that are permuted by the action of the diffeomorphism f. Note that, by transitivity, this permutation must be a cyclic permutation (that is, $\Lambda = \bigcup_{i=1}^k f^i(\Lambda_1)$). In some sense, we can say that each compact subset Λ_i as above is a "copy" of the other ones, and they exhibit the "same" dynamics for f^k. Hence the study of the dynamics in Λ can be restricted to the study of one of these smaller pieces.

Definition 7.1 (Spectral decomposition) We say that a transitive compact invariant set Λ admits a *spectral decomposition* if there exist compact sets $\Lambda_1, \Lambda_2, .., \Lambda_k$ satisfying:

1. $\Lambda = \bigcup_{i=1}^k \Lambda_i$.
2. There is a cyclic permutation $\sigma : \{1, ..., k\} \circlearrowleft$ such that $f(\Lambda_i) = \Lambda_{\sigma(i)}$ for all $i \in \{1, ..., k\}$. In particular, Λ_i is periodic with period k.
3. $\Lambda_i \cap \Lambda_j = \emptyset$ for all $i \neq j$ in $\{1, ..., k\}$.
4. For every $i \in \{1, ..., k\}$, Λ_i is topologically mixing for the map f^k.

We call the sets Λ_i the *components* or the *basic pieces* of Λ.

Remark 7.2 As the permutation in item (2) is cyclic, the period of any periodic point in Λ is a multiple of the number k of components of Λ.

Definition 7.3 Let Λ be an s-minimal set. We say that d is the *minimal constant* of Λ if d is the smallest number verifying the s-minimality definition for Λ.

The main result in this section is the following theorem and its robust version for robustly transitive attractors in Theorem 7.8.

Theorem 7.4 *Let and $\Lambda = H(p, f)$ be an s-minimal (or u-minimal) (s, c, u)-partially hyperbolic attractor with minimal constant d. Then Λ admits a unique spectral decomposition with exactly d components.*

To prove this theorem we need some auxiliary lemmas.

Lemma 7.5 *Let $\Lambda = H(p, f)$ be an s-minimal attractor with minimal constant d. Let $x \in \Lambda$ and $k > 1$ be such that*

$$\bigcup_{i=1}^{k} \overline{\mathcal{F}^s_\Lambda(f^i(x))} = \Lambda.$$

Then $k \geq d$.

Proof: Fix $y \in \Lambda$. From s-minimality, we get that

$$\bigcup_{i=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^i(y))} = \Lambda.$$

Then there is some $m \in \{1, ..., d\}$ such that $x \in \overline{\mathcal{F}^s_\Lambda(f^m(y))}$. It follows from the continuity of the foliation that $\mathcal{F}^s(x) \subset \overline{\mathcal{F}^s(f^m(y))}$. Denoting $w = f^m(y)$ and by Lemma 5.11, we get that:

$$\Lambda = \bigcup_{i=1}^{k} \overline{\mathcal{F}^s_\Lambda(f^i(x))} \subset \bigcup_{i=1}^{k} \overline{\mathcal{F}^s_\Lambda(f^{m+i}(y))} = \bigcup_{i=1}^{k} \overline{\mathcal{F}^s_\Lambda(f^i(w))} = \Lambda.$$

Since it holds for every $y \in \Lambda$ (and then for every $w \in \Lambda$), we get that k is a constant for which the s-minimality holds. Then, from the definition of minimal constant, we get that $k \geq d$. ■

Lemma 7.6 *Let Λ be as in Lemma 7.5. For every $x \in \Lambda$ the sequence of sets $\{\overline{\mathcal{F}^s_\Lambda(f^n(x))}\}_{n=1}^{d}$ is pairwise disjoint.*

Proof: Suppose that $\overline{\mathcal{F}^s_\Lambda(f^i(x))} \cap \overline{\mathcal{F}^s_\Lambda(f^j(x))} \neq \emptyset$ for some $i < j$ in $\{1, ..., d\}$ and denote $m = j - i$. Take a point z in this intersection, and observe that, by Lemma 5.11, the set $\mathcal{F}^s_\Lambda(z)$ is contained in this intersection. Since $\mathcal{F}^s_\Lambda(z) \subset \overline{\mathcal{F}^s_\Lambda(f^j(x))}$, we have that

$$\bigcup_{n=1}^{m} \overline{\mathcal{F}^s_\Lambda(f^n(z))} \subset \bigcup_{n=j+1}^{2j-i} \overline{\mathcal{F}^s_\Lambda(f^n(x))}. \tag{7.0.1}$$

Since $\mathcal{F}^s_\Lambda(z) \subset \overline{\mathcal{F}^s_\Lambda(f^i(x))}$, we have that $\mathcal{F}^s_\Lambda(f^m(z)) \subset \overline{\mathcal{F}^s_\Lambda(f^j(x))}$, and consequently we obtain

$$\bigcup_{n=m+1}^{d} \overline{\mathcal{F}^s_\Lambda(f^n(z))} \subset \bigcup_{n=j+1}^{d+i} \overline{\mathcal{F}^s_\Lambda(f^n(x))}. \tag{7.0.2}$$

Denoting $r = max\{2j-i, d+i\}$, $w = f^j(x)$, and putting together the deinition of w and Equations (7.0.1) and (7.0.2), we conclude that

$$\Lambda = \bigcup_{n=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^n(z))} \subset \bigcup_{n=j+1}^{r} \overline{\mathcal{F}^s_\Lambda(f^n(x))} = \bigcup_{n=1}^{r-j} \overline{\mathcal{F}^s_\Lambda(f^n(w))}.$$

This contradicts lemma 7.5, since $r - j = max\{m, d-m\} < d$. ■

ProofProof of Theorem 7.4.: We have to prove items (1),(2),(3) and (4) of Definition 7.1 with $k = d$.

Take some $x \in \Lambda$ and set $\Lambda_i = f^i(\overline{\mathcal{F}^s_\Lambda(x)})$ for $i \in \{1, \ldots, d\}$. Items (1) of Definition 7.1 is an immediate consequence of s-minimality.

For item (2), set $\sigma(i) = i+1$ for $1 \le i < d$ and $\sigma(d) = 1$. It is clear that $f(\Lambda_i) = \Lambda_{i+1} = \Lambda_{\sigma(i)}$ for all $1 \le i < d$. So we only have to prove that $f(\Lambda_d) = \Lambda_{\sigma(d)} = \Lambda_1$.

Considering p and $f(p)$ in Lemma 7.6, we obtain

$$\Lambda = \bigcup_{n=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^n(p))} = \bigcup_{n=2}^{d+1} \overline{\mathcal{F}^s_\Lambda(f^n(p))}.$$

Since these unions consist of pairwise disjoint sets, we conclude that $\overline{\mathcal{F}^s_\Lambda(f(p))} = \overline{\mathcal{F}^s_\Lambda(f^{d+1}(p))}$, which means that $\Lambda_1 = f(\Lambda_d)$.

Item (3) is just Lemma 7.6.

For item (4), fix $i \in \{1, \ldots, d\}$ and two relative open sets A, B of Λ_i. Consider a hyperbolic periodic point $p_a \in A$. From s-minimality, there is some $j \in \{0, \ldots, d-1\}$ such that $f^j(\mathcal{F}^s(p_a)) \cap B \neq \emptyset$. Since A and B lie in the same component, j must be zero. Then, for every $r > 0$ there is k (a multiple of the period of p_a) sufficiently large so that

$$f^{-k}(\mathcal{F}^s_r(p_a)) \cap B \neq \emptyset.$$

By the cyclic permutation of item (2), we also have that

$$f^{-k-n.d}(\mathcal{F}^s_r(p_a)) \cap B \neq \emptyset,$$

for every $n \in \mathbb{N}$. Since k is a multiple of the period of p_a and by Remark 7.2, we can write

$$f^{d.(L-n)}(\mathcal{F}^s_r(p_a)) \cap B \neq \emptyset,$$

for some $L \in \mathbb{Z}$. This means that the map f^d is mixing on Λ_i. ■

Theorem 7.7 *Let Λ be a robustly s-minimal attractor with minimal constant d for some $f \in \mathcal{R}$ with a hyperbolic periodic point p. Then there is a neighborhood $\mathcal{U}$ of f such that the minimal constant of every $g \in \mathcal{U}$ is also d.*

The same is true for generically s-minimal sets, replacing the neighborhood $\mathcal{U}$ by a locally generic neighborhood $\mathcal{U} \cap \mathcal{R}$ (where $\mathcal{R}$ is a C^1-residual subset of $\mathrm{Diff}^1(M)$*).*

Proof: Let m be the period of the hyperbolic periodic point p. By Theorem 7.4 and Remark 7.2, there is $n \in \mathbb{N}$ such that $m = n \cdot d$. From s-minimality, we get that $\Lambda = \bigcup_{n=1}^{d} \overline{\mathcal{F}^s_\Lambda(f^n(p))}$. By item (2) in definition 7.1, with $k = d$, for every $i \in \{1, \dots, d\}$ the set Λ_i coincides with the following sets

$$\overline{\mathcal{F}^s_\Lambda(f^i(p))} = \overline{\mathcal{F}^s_\Lambda(f^{d+i}(p))} = \cdots = \overline{\mathcal{F}^s_\Lambda(f^{(n-1)d+i}(p))}. \tag{7.0.3}$$

This equation implies that $\mathcal{F}^s(f^i(p))$ intersects transversally the unstable manifold of $f^{d+i}(p), f^{2d+i}(p), \dots, f^{(n-1)d}(p)$, which is a robust property. By the λ-lemma, it holds that

$$\overline{\mathcal{F}^s_{\Lambda_g}(g^i(p))} = \overline{\mathcal{F}^s_{\Lambda_g}(g^{d+i}(p))} = \cdots = \overline{\mathcal{F}^s_{\Lambda_g}(g^{(n-1)d+i}(p))}. \tag{7.0.4}$$

This shows that the number of pieces in the spectral decomposition of g in a small neighborhood of f cannot increase.

On the other hand, by Remark 2.2, the pairwise disjoint compact sets $\{\Lambda_i\}_{i=1}^d$ admit upper semicontinuations for any diffeomorphism g sufficiently close to f, and the cyclic permutation given by f induces a cyclic permutation given by g on these continuations. Hence the number of components of $\Lambda_g(U)$ must be bigger or equal to d.

As a conclusion, the spectral decomposition of g has exactly d components. ■

Theorem 7.8 *Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be a $(s,1,u)$-partially hyperbolic robustly transitive attractor. Then $\Lambda_f(U)$ has a robust spectral decomposition: every g*

in a small neighborhood of f has a spectral decomposition whose pieces are the continuations of the pieces of the spectral decomposition of Λ_f.

Proof: By Corollary 5.17 and Theorem 7.7, we can assume that f is robustly s-minimal with a minimal constant d defined for all g in a neighborhood of f. By Theorem 7.4, we obtain the spectral decomposition of $\Lambda_g(U)$ into d pairwise disjoint basic pieces. Finally, by Corollary 6.4, the d components of the espectral decomposition of $\Lambda_g(U)$ must be the continuation of the d components of the spectral decomposition of f. ■